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THE VARIANCE RATIO TEST: AN ANALYSIS OF SIZE AND POWER BASED ON A CONTINUOUS-TIME ASYMPTOTIC FRAMEWORK

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We consider the statistical properties of the variance ratio statistic in the context of testing for market efficiency defined by uncorrelated returns. The statistic is then the ratio of the variance of K-period returns to the variance of one-period returns scaled by K. We use a continuous-time asymptotic framework whereby we let the sample size increase to infinity keeping the span of the data fixed. We also let the aggregation parameter K increase such that $K/T \to \kappa$ as T, the sample size, increases to infinity. We consider the limit of the statistic under the null hypothesis and under three alternative hypotheses that have been popular in the finance literature. Our analysis permits us to address size and power issues with respect to κ and the sampling interval used. Our theoretical and simulation results show that power is initially increasing as κ increases but then decreases with further increases in κ. This shows that for any given alternative there exists a value of K relative to T that will maximize power. We thus investigate the properties of a test that is the maximal value of the variance ratio over a range of possible values for K. The importance of the trimming to define this range is highlighted.

1. INTRODUCTION

Since the seminal work of Bachelier (1900) and Fama (1965), the random walk hypothesis has been an integral part of theories pertaining to financial time series. In particular, this hypothesis allows a formal statistical framework to model the concept of market efficiency in the sense that the best predictor of future prices is current ones (Fama, 1970, 1991). Many studies have consid-

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ered testing the null hypothesis of a random walk for prices against a variety of alternative hypotheses. Some specify a time series representation different from the random walk (e.g., a stationary autoregressive process or the sum of a permanent and transitory component; see Fama and French, 1988a; Poterba and Summers, 1988; Lo and Mackinlay, 1988). Others attempt to assess whether some regressors have predictive power for returns at some horizon (e.g., lagged returns, interest rate, or the dividend-price ratio; Hansen and Hodrick, 1980; Fama and French, 1988a, 1988b). A common conclusion is that the market efficiency hypothesis is rejected when using long horizon returns, which implies a mean-reverting behavior for prices. In particular, evidence has been put forward to the effect that long horizon returns (3 to 10 years) are negatively correlated (Fama and French, 1988a; Poterba and Summers, 1988) and that the dividend-price ratio has a positive effect on excess returns (Rozeff, 1984; Shiller, 1984; Campbell and Shiller, 1988). According to Fama and French (1988b), this effect is weak for returns computed over short horizons (1 to 3 months) but substantial for long horizon returns (2 to 4 years).

When using statistics based on K-period returns, simulations have shown that the standard asymptotic normal distribution provides a poor approximation to the exact finite sample distribution, which leads to tests with distorted sizes. Another concern is the power of the tests, especially with respect to the aggregation parameter K. Richardson and Stock (1989) have proposed, as a solution to the size problem, an asymptotic framework whereby the aggregation parameter of returns, K, is a fixed fraction of total sample size such that $K/T \to \kappa$ as $T \to \infty$. The limit distribution for the autocorrelation coefficient of K-period returns and the variance ratio are then nonstandard and functions of Weiner processes. The quality of the finite sample approximations provided by this asymptotic framework is good according to simulation results reported in Richardson and Stock (1989).

In this paper, we provide further analyses of the variance ratio statistic using the asymptotic framework with $K/T \to \kappa$. Our approach is to posit a continuous-time process of interest, derive its discrete-time representation, and take the limit as the sample size T increases keeping the span of the data fixed (i.e., letting the sampling interval tend to zero at rate T).

The variance ratio test has been the object of several studies in the finance and econometrics literature. The idea behind this test is that, with uncorrelated returns, the sum of the variances of equidistant K-period returns should, in population, be equal to K times the variance of one-period returns. A natural way to construct a test is then to replace population variances by sampling variances. Let R_{th} denote returns computed over one period of length h, defining the sampling interval used, and $R_{th}(K) = \sum_{i=0}^{K-1} R_{(t-i)h}$ denote K-period returns; the variance ratio statistic is defined by

$$VR(K) = \frac{1}{K} \operatorname{var}(R_{th}(K))/\operatorname{var}(R_{th}), \tag{1}$$

where $\text{var}(R_{th}(K)) = T^{-1}\sum_{t=K-1}^{T}(R_{th}(K) - K\hat{\mu})^2$ and $\text{var}(R_{th}) = T^{-1}\sum_{t=1}^{T}(R_{th} - \hat{\mu})^2$ with $\hat{\mu} = T^{-1}\sum_{t=1}^{T} R_{th}$. It was applied to U.S. real gross national product (GNP) by Cochrane (1986) and to financial assets by Poterba and Summers (1988), Lo and Mackinlay (1989), and Faust (1988, 1992).

Lo and Mackinlay (1988, 1989) derive the basic asymptotic distributions for a fixed K and provide an extensive simulation analysis of size and power. They show important size distortions, and their results illustrate the fact that power may be a nonmonotonic function of K. Richardson and Smith (1991, 1994) analyze, among other things, asymptotic power using the approximate slope function of Bahadur (1960) and Geweke (1981). A similar approximate slope analysis is provided by Campbell (2001) in a related context (regression-based tests). Faust (1992) shows that the variance ratio statistic is approximately optimal for some class of alternatives (this class varying with K). Daniel (2001) uses a different asymptotic framework to analyze power whereby returns are modeled as being locally uncorrelated (i.e., returns are independent and identically distributed [i.i.d.] plus a stationary autoregressive moving average [ARMA] process whose importance vanishes at rate $T^{1/4}$). As mentioned earlier, Richardson and Stock (1989) use an asymptotic framework whereby $K/T \to \kappa$. They show that the resulting asymptotic distribution provides a much improved approximation to the finite sample distribution. Deo and Richardson (2003) show that under this asymptotic framework, the variance ratio test is inconsistent. We shall comment on some of these contributions throughout the paper. The bottom line is that our continuous-time asymptotic framework retains the advantages of the analysis of Richardson and Stock (1989) for size but offers a much improved treatment of power.

The rest of this paper is structured as follows. Section 2 describes the limit distribution of the variance ratio statistic under the null hypothesis of market efficiency defined by uncorrelated returns. In Section 3, we consider three alternative hypotheses of interest: (a) with the dividend-price ratio as a predictor of returns (modeled as a near-unit-root process); (b) with trend-stationary prices; and (c) with prices as the sum of a permanent and transitory component. The limit distribution of the statistic is derived for each case. Section 4 presents simulation experiments to investigate the following issues: the adequacy of our asymptotic distributions as approximations to the finite sample distributions (under both the null and alternative hypotheses); and what features influence the power of the test. Our results show, in particular, that the power of the variance ratio statistic initially increases with an increase in the aggregation parameter κ but subsequently decreases as κ is increased further. This has important practical implications because it indicates that for a given alternative hypothesis there is a value of K relative to T that maximizes power. Section 5 investigates the properties of a test that is the maximal value of the variance ratio over a range of possible values for K. Critical values are provided, and we discuss how the trimming to define this range affects power. Section 6 presents concluding remarks, and an Appendix gives technical derivations.

2. LIMIT DISTRIBUTION UNDER THE NULL HYPOTHESIS

Under the null hypothesis, we consider the following continuous-time stochastic process for $P(t)$, the logarithm of the price of an asset or a portfolio at time t:

$$dP(t) = \sigma\, dW(t), \tag{2}$$

with $t \in [0, N]$ and $P(0) = O_p(1)$, an arbitrary constant or a random variable. The discrete-time representation of the process for a sampling interval of length h is given by

$$P_{th} = P_{(t-1)h} + \epsilon_{th}, \qquad t = 1, \ldots, T \equiv N/h,$$

where N is the span of the data, h is the sampling interval, and ϵ_{th} is i.i.d. $N(0, h\sigma^2)$. We define returns over a period of length h by $R_{th} = P_{th} - P_{(t-1)h}$. Under the null hypothesis, the discrete-time representation for returns is then

$$H_0 : R_{th} = \epsilon_{th}, \qquad t = 1, \ldots, T \equiv N/h. \tag{3}$$

Note that the Gaussian assumption on the errors that follows from taking a discrete-time approximation to a continuous-time model is not restrictive. The same asymptotic results hold allowing a general class of processes for the errors of the discrete-time model. What is needed are conditions on the discrete-time errors such that the results stated in Lemma A.1 in the Appendix hold. Such conditions can allow nonnormal processes with some forms of heteroskedasticity. This holds true under both the null and the class of alternative hypotheses to be considered later.

2.1. Case with *K* Fixed

When K is fixed, Lo and Mackinlay (1988) and Faust (1988) showed that the limiting distribution under the null hypothesis (3) is

$$T^{1/2}(VR(K) - 1) \Rightarrow N(0, 2(2K-1)(K-1)/3K). \tag{4}$$

Lo and Mackinlay (1989) considered the adequacy of this normal asymptotic distribution as an approximation to the finite sample distribution, in particular for the case where the data are generated by (3). They showed the approximation to be adequate when K is small and T is large but less so when K is large. They also showed that the variance ratio test can have better power than other tests under several alternatives (log-prices following an AR(1) process, log-prices having both a permanent and a transitory component, returns following an AR(1) process). Faust (1988) notes, however, that these power gains are seriously undermarked by the presence of substantial size distortions that increase as K increases. For example, when $T = 732$ and for a nominal size of 5%, the exact size of the test is 1.6% for $K = 48$, 0.06% for $K = 72$, and 0.0% for $K = 120$.

2.2. Case with $K \to \infty$ and $K/T \to \kappa$

The simulation results discussed previously suggest that the asymptotic normal approximation is inadequate when K is rather large relative to the sample size T. Hence, we consider the alternative asymptotic framework whereby K increases to infinity and the ratio K/T tends to a limit κ $(0 < \kappa < 1)$. The limit distribution of the statistic $VR(K)$ under the null hypothesis is given in the following proposition, proved in the Appendix.

PROPOSITION 1. *If* $K/T \to \kappa$ *and* $K \to \infty$ *as* $T \to \infty$ *with* $0 < \kappa < 1$ *and* N *fixed, then under the null hypothesis (3), we have*

$$VR(K) \Rightarrow \frac{1}{\kappa} \int_{\kappa}^{1} [W(r) - W(r-\kappa) - \kappa W(1)]^2 \, dr, \tag{5}$$

with $W(r)$ *a standard Wiener process.*

The limit distribution (5) is the same as that derived by Richardson and Stock (1989), who used a fixed h asymptotic framework with $K/T \to \kappa$ as $T \to \infty$. Note that the limiting distribution depends on the nuisance parameter κ.

3. LIMIT DISTRIBUTION UNDER SEVERAL ALTERNATIVES

3.1. Alternative H_1: Dividend-Price Ratio as a Predictor

We consider two continuous-time stochastic processes, $P(t)$ denoting the logarithm of the price of an asset or a portfolio and $X(t)$ a variable such as the dividend-price ratio. We let $Z(t) = (P(t), X(t))'$ and assume that $Z(t)$ is generated by the diffusion process

$$dZ(t) = AZ(t)\, dt + dW(t), \tag{6}$$

with $t \in [0, N]$ and $Z(0) = (P_0, X_0)' = O_p(1)$,

$$A = \begin{bmatrix} 0 & \beta \\ 0 & \gamma \end{bmatrix},$$

and where $W = (W_1, W_2)'$ is a two-dimensional standard Wiener process with covariance

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & \tau \end{bmatrix}. \tag{7}$$

The solution to the stochastic differential equation (6) is (e.g., Arnold, 1974)

$$Z(t) = \exp(tA)Z(0) + \int_0^t \exp((t-s)A)\, dW(s), \tag{8}$$

with

$$\exp(uA) = \begin{bmatrix} 1 & \frac{\beta}{\gamma}(\exp(\gamma u) - 1) \\ 0 & \exp(\gamma u) \end{bmatrix}.$$

Assuming that $Z(t)$ is observed over the time interval $[0, N]$, and defining the sampling interval h by $Th = N$, we can obtain the exact discrete-time representation Z_{th} of $Z(t)$ that is given by the following autoregressive process of order one:

$$Z_{th} = \exp(hA) Z_{(t-1)h} + u_{th}, \qquad t = 1, \ldots, T, \tag{9}$$

where $Z_{th} = (P_{th}, X_{th})'$, $Z_{0h} = Z(0)$, and

$$u_{th} = \int_{(t-1)h}^{th} \exp((th - r)A)\, dW(r).$$

The random component u_{th} is i.i.d. $N(0, \Omega_h)$ with

$$\Omega_h = \int_0^h \exp(sA) \Sigma \exp(sA)'\, ds \equiv \Omega_h = \begin{bmatrix} \Omega_h^{11} & \Omega_h^{12} \\ \Omega_h^{21} & \Omega_h^{22} \end{bmatrix},$$

where

$$\Omega_h^{11} = h + 2\rho h \frac{\beta}{\gamma}\left(\frac{\exp(\gamma h) - 1}{\gamma h} - 1\right)$$
$$+ \tau h \frac{\beta^2}{\gamma^2}\left(\frac{\exp(2\gamma h) - 4\exp(\gamma h) + 3}{2\gamma h} + 1\right),$$
$$\Omega_h^{12} = \Omega_h^{21} = \rho \frac{\exp(\gamma h) - 1}{\gamma} + \frac{\tau\beta}{2}\left(\frac{\exp(\gamma h) - 1}{\gamma}\right)^2,$$
$$\Omega_h^{22} = \tau \frac{\exp(2\gamma h) - 1}{2\gamma}.$$

As before, returns over a sampling interval of length h are given by $R_{th} = P_{th} - P_{(t-1)h}$. Using the notation $u_{th} = (\epsilon_{th}, v_{th})'$ and $\alpha_h = \beta(\exp(\gamma h) - 1)/\gamma$, we obtain from (9) the following discrete-time model for returns and the dividend-price ratio:

$$\begin{aligned} R_{th} &= \alpha_h X_{(t-1)h} + \epsilon_{th}, \\ X_{th} &= \exp(\gamma h) X_{(t-1)h} + v_{th} \end{aligned} \tag{10}$$

for $t = 1, \ldots, T \equiv N/h$ and where $u_{th} = (\epsilon_{th}, v_{th})' \sim$ i.i.d. $N(0, \Omega_h)$.

For a fixed sampling interval, the system (10) implies that the univariate process for returns, R_{th}, is an ARMA(1,1), consistent with the idea that asset prices have permanent and transitory components (e.g., Poterba and Summers, 1988; Campbell, 2001). It also implies that conditional on information available at time t, I_{th}, expected returns are given by $E(R_{(t+1)h}|I_{th}) = \alpha_h X_{th}$. Accordingly, expectations of future returns are affected by the dividend-price ratio. Letting $c = \gamma N$ and $g = \beta N$, we can write $\alpha_h = g(\exp(c/T) - 1)/c \simeq g/T$. Hence, in the asymptotic framework where $T \to \infty$ with N fixed, we can interpret α_h as a sequence of local alternatives with noncentrality parameter g. Under these specifications, the limit of the variance ratio statistic when $K \to \infty$ and $K/T \to \kappa$ as $T \to \infty$ is given in the following proposition, proved in the Appendix.

PROPOSITION 2. *If $K \to \infty$ and $K/T \to \kappa$ with $0 < \kappa < 1$, when $T \to \infty$ with N fixed, we have, under the alternative H_1,*

$$VR(K) \Rightarrow \frac{1}{\kappa}\int_{\kappa}^{1}\left\{g\int_{r-\kappa}^{r}[\sqrt{\tau}J_2(s) + \exp(cs)X_0^*]\,ds + W_{12}(r) - W_{12}(r-\kappa)\right.$$
$$\left. - \kappa\left[g\int_0^1[\sqrt{\tau}J_2(s) + \exp(cs)X_0^*]\,ds + W_{12}(1)\right]\right\}^2 dr \qquad \textbf{(11)}$$

with $g = \beta N$, $X_0^* = X_0/N^{1/2}$, and $W_{12}(r) = \sqrt{1-\delta^2}W_1(r) + \delta W_2(r)$. Also, $W_1(r)$ and $W_2(r)$ are independent Wiener processes, $\delta = \rho/\sqrt{\tau}$, and $J_2(s) = \int_0^s \exp(c(s-u))\,dW_2(u)$.

We consider, as in Lo and Mackinlay (1989), the two-sided critical region defined by $\{VR(K) < \lambda_1$ or $VR(K) > \lambda_2\}$ where λ_1 and λ_2 are constants. Denoting by $\lambda_{\alpha_1}(\kappa)$ and $\lambda_{1-\alpha_2}(\kappa)$ the quantiles of order α_1 and $1-\alpha_2$ of the limit distribution (5), the asymptotic power of a two-sided test with size μ $(\alpha_1 + \alpha_2 = \mu)$ is given by, for any $g \geq 0$,

$$\lim_{T,K\to\infty} P^1_{T,K}(g,\kappa,c) = \lim_{h\to 0}\Pr_{H_1}\{VR(K) < \lambda_{\alpha_1}(\kappa) \quad \text{or} \quad VR(K) > \lambda_{1-\alpha_2}(\kappa)\}$$
$$= \Pr\{L_1(g,\kappa,c,\delta) < \lambda_{\alpha_1}(\kappa) \quad \text{or} \quad L_1(g,\kappa,c,\delta) > \lambda_{1-\alpha_2}(\kappa)\},$$

where $L_1(g,\kappa,c,\delta)$ is the limit distribution (11).

3.2. Alternative H_2: Mean-Reverting Prices

As a second alternative hypothesis of interest, we follow Shiller and Perron (1985), among others, and suppose that the logarithm of prices is a stationary Ornstein–Ühlenbeck process:

$$dP(t) = \gamma(P(t) - \eta)\,dt + \sigma\,dW(t),$$

with $\gamma < 0$, $P(0) = P_0$, and $t \in [0,N]$. We suppose, for simplicity, that $\eta = 0$. The discrete-time representation is given by the following AR(1) process, for $t = 1,\ldots,T \equiv N/h$:

$$P_{th} = \exp(\gamma h)P_{(t-1)h} + \epsilon_{th}, \tag{12}$$

with $P_{0h} = P_0$ and where $\epsilon_{th} \sim$ i.i.d. $N(0, a(h))$ with $a(h) = \sigma^2(\exp(2\gamma h) - 1)/2\gamma$.

Returns being defined as $R_{th} = P_{th} - P_{(t-1)h}$, the limit distribution, as $T \to \infty$ with N fixed, of the variance ratio statistic is given by, under this alternative hypothesis,

$$VR(K) \Rightarrow \frac{1}{\kappa}\int_{\kappa}^{1}\left[c\int_{r-\kappa}^{r}[J_c(s) + \exp(cs)P_0^*]\,ds + W(r) - W(r-\kappa)\right.$$
$$\left. - \kappa\left(c\int_{0}^{1}[J_c(s) + \exp(cs)P_0^*]\,ds + W(1)\right)\right]^2 dr, \tag{13}$$

where $c = \gamma N$, $P_0^* = P_0/[N\sigma^2]^{1/2}$, and $J_c(s) = \int_0^s \exp(c(s-u))\,dW(u)$. Setting $\gamma = 0$, it is easily seen that the same null limiting distribution obtains.

3.3. Alternative H_3: Prices as the Sum of Permanent and Transitory Components

As a third alternative hypothesis of interest, we suppose that the logarithm of prices is represented by the sum of two processes, namely, a permanent component $P(t)^b$ and a transitory component $P(t)^a$. These processes are specified as follows, for $t \in [0, N]$:

$$dP(t)^a = \gamma P(t)^a\,dt + \sigma^a\,dW_a(t),$$

$$dp(t)^b = \alpha p(t)^b\,dt + \sigma^b p(t)^b\,dW_b(t),$$

where $p(t)^b = \exp(P(t)^b)$ is the level of the permanent component of prices and $W_a(t)$ and $W_b(t)$ are two independent Wiener processes. The initial values are $P(0)^a = P_0^a$ and $P(0)^b = P_0^b$. It is assumed that $\gamma < 0$ and, hence, that the transitory component, $P(t)^a$, is a stationary Ornstein–Ühlenbeck process. The permanent component, $P(t)^b$, is a geometric Brownian motion. The discrete-time representations are given by

$$\begin{aligned} P_{th}^a &= \exp(\gamma h)P_{(t-1)h}^a + u_{th}, \\ P_{th}^b &= P_{(t-1)h}^b + (\alpha - (\sigma^b)^2/2)h + v_{th}, \end{aligned} \tag{14}$$

with $P_{0h}^a = P_0^a$, $P_{0h}^b = P_0^b$, and where $(u_{th}, v_{th})' \sim$ i.i.d. $N(0, \Omega_h)$ with

$$\Omega_h = \begin{bmatrix} (\sigma^a)^2(\exp(2\gamma h) - 1)/2\gamma & 0 \\ 0 & (\sigma^b)^2 h \end{bmatrix}.$$

We suppose for simplicity that $\alpha = (\sigma^b)^2/2$ so that the permanent component has no drift. Returns are defined by $R_{th} = P_{th} - P_{(t-1)h}$ where $P_{th} = P_{th}^a + P_{th}^b$. The stochastic process for returns is then an ARMA(1,1) similar to that considered by Summers (1986), Poterba and Summers (1988), and Fama and French (1988a).

It is straightforward to show that the limit distribution of the variance ratio statistic under H_3, as $T \to \infty$ with N fixed, is given by

$$VR(K) \Rightarrow \frac{1}{\kappa(1+\tau^2)} \int_\kappa^1 \left[c \int_{r-\kappa}^{r} [J_c^a(s) + \exp(cs)P_0^{a*}]\, ds + W(r) - W(r-\kappa) \right. \\ \left. - \kappa \left(c \int_0^1 [J_c^a(s) + \exp(cs)P_0^{a*}]\, ds + W(1) \right) \right]^2 dr \tag{15}$$

with $c = \gamma N$, $P_0^{a*} = P_0^a/[N\sigma_a^2]^{1/2}$, $\tau = \sigma^b/\sigma^a$, $J_c^a(s) = \int_0^s \exp(c(s-u))\, dW_a(u)$, and $W(r) = W_a(r) + \tau W_b(r)$. Under H_0, $\gamma = 0$, and we recover the result of Proposition 1.

Under the alternatives H_2 and H_3, the critical regions are defined as for H_1. The asymptotic power of a two-sided test with significance level μ is given by, for $i = 2, 3$,

$$\lim_{T,K\to\infty} P_{T,K}^i(\kappa, c, \tau) = \Pr\{L_i(\kappa, c, \tau) < \lambda_{\alpha_1}(\kappa) \text{ or } L_i(\kappa, c, \tau) > \lambda_{1-\alpha_2}(\kappa)\}$$

with $c \leq 0$, $\lambda_{\alpha_1}(\kappa)$, and $\lambda_{1-\alpha_2}(\kappa)$ being the quantiles of order α_1 and $1-\alpha_2$ ($\alpha_1 + \alpha_2 = \mu$) of the limit distribution (5) and $L_i(\kappa, c, \tau)$, $i = 2, 3$, the limit distribution under the relevant alternative H_i, i.e., either (13) or (15). Under the alternatives H_2 and H_3, power depends on κ and c and the normalized initial value of the transitory component P_0^{a*}. Under H_3, it also depends on τ, the square root of the limit, as h converges to 0, of the relative variance of the permanent and transitory components. As τ increases, the permanent component becomes more important and the power approaches the size of the test and vice versa.

3.4. Remarks on Consistency

In the asymptotic distributions presented previously, the relevant noncentrality parameters that measure the distance between the null and the alternative hypotheses are $g = \beta N$ for H_1 and $c = \gamma N$ for H_2 and H_3. Hence, the tests will be consistent against these alternatives provided the span of the data N increases as the sample size increases. Increasing the sample size is not enough to ensure consistency of the test (see Perron, 1991). This is in contrast to the result of Deo and Richardson (2003), who show that the variance ratio statistic is inconsistent in the standard $K/T \to \kappa$ asymptotic framework. The main reason for the discrepancy is the ad hoc nature of the standard $K/T \to \kappa$ asymptotic framework. As a result, the noncentrality parameter that measures the distance between the null and the alternative hypothesis is held fixed as the sample size increases, irrespective of the span of the data or the sampling interval. Our continuous-time asymptotic framework directly links the noncentrality parameter as a function of the underlying continuous-time parameters and the span of the data. Hence, it provides a more coherent framework to analyze power. In light of this, the simulation results of Deo and Richardson (2003) can be interpreted as pertaining to the asymptotic power when T increases and N, the span of the data, is held fixed. Hence, it is not surprising that power does not increase to one for fixed alternatives.

4. SIMULATION EXPERIMENTS

The aim is first to present results pertaining to the power function for various sample sizes and parameters and show that the asymptotic power functions provide good approximations in moderate-sized samples. We then highlight the effect of the aggregation parameter κ on the asymptotic power. We do not discuss the adequacy of the asymptotic distribution (5) under the null hypothesis. This was studied by Richardson and Stock (1989), who concluded that it indeed provides a good approximation, far superior to the fixed K asymptotic distribution examined by Lo and Mackinlay (1989); see Section 2.1.

We first consider the case where the initial values are set to zero. The effects of nonzero values are assessed in Section 4.4. The exact distributions under the null hypothesis are obtained for different values of T using 10,000 simulations of the process (3). The exact power functions are then computed simulating the exact distributions under the various alternative hypotheses. The limiting distributions are also obtained using simulations; quantities such as $J_2(r)$ and $W_{12}(r)$ are approximated by $T^{-1/2}X_{[Tr]-1}$ and $T^{-1/2}\sum_{i=1}^{[Tr]}(\sqrt{1-\delta^2}u_i + \delta v_i)$, respectively. The various integrals are constructed by the appropriate normalized sums with $T = 1{,}000$, and $X_t = \exp(c/T)X_{t-1} + v_t$ where u_t and v_t are drawn independently from a $N(0,1)$. Other quantities are simulated in a similar fashion.

4.1. Finite Sample and Asymptotic Power

We now consider the finite sample power under the three alternatives described before. To that effect, we use the following values of N (the total horizon of the data) and T (the total number of observations) with $h = N/T$ defining the sampling interval: $N, T = 8, 16, 32, 64, 128, 256, 512, 1{,}024, 2{,}048$, and ∞. The results are presented in Tables 1–3 for two-sided tests of nominal size 5% and values of the aggregation parameter $\kappa = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}$, and $\frac{1}{2}$. The values selected for the various parameters under the alternative hypotheses are as follows. For H_1, $\delta = 0$, $\beta = 0.1$ and $\gamma = -0.02$; hence the regressor is mean-reverting and affects returns in a positive way.[1] For H_2, $\gamma = -0.2$, and prices are accordingly stationary. For H_3, the transitory component is again generated with $\gamma = -0.2$ and $\tau = \frac{1}{2}$ so that the noise of the transitory components dominates. The values

Table 1. Power of $VR(K)$ against H_1: $\delta = 0.0$, $\beta = 0.1$, $\gamma = -0.02$

T	8	16	32	64	128	256	512	1,024	2,048	∞
$\kappa = \frac{1}{16}$										
$N =$ 8			0.053	0.048	0.049	0.049	0.055	0.049	0.052	0.049
16			0.053	0.049	0.053	0.050	0.049	0.050	0.064	0.049
32			0.068	0.071	0.070	0.068	0.066	0.066	0.075	0.069
64			0.115	0.136	0.158	0.152	0.155	0.160	0.160	0.162
128			0.330	0.395	0.455	0.451	0.461	0.466	0.476	0.483
256			0.656	0.779	0.832	0.846	0.860	0.860	0.870	0.870
512			0.858	0.958	0.980	0.987	0.991	0.992	0.994	0.994
cv1			0.613	0.526	0.490	0.475	0.470	0.460	0.469	0.463
cv2			1.269	1.382	1.418	1.459	1.473	1.478	1.470	1.476
$\kappa = \frac{1}{8}$										
$N =$ 8		0.043	0.049	0.054	0.053	0.049	0.053	0.058	0.051	0.057
16		0.049	0.050	0.060	0.056	0.056	0.058	0.057	0.052	0.066
32		0.059	0.068	0.081	0.084	0.078	0.080	0.083	0.076	0.086
64		0.110	0.144	0.174	0.189	0.187	0.194	0.196	0.192	0.218
128		0.264	0.383	0.455	0.490	0.501	0.516	0.518	0.513	0.524
256		0.453	0.674	0.785	0.828	0.838	0.857	0.860	0.853	0.875
512		0.496	0.800	0.925	0.963	0.970	0.977	0.981	0.979	0.984
cv1		0.435	0.365	0.330	0.309	0.295	0.294	0.293	0.293	0.292
cv2		1.344	1.484	1.521	1.533	1.569	1.564	1.557	1.594	1.539
$\kappa = \frac{1}{4}$										
$N =$ 8	0.055	0.054	0.049	0.050	0.047	0.058	0.054	0.058	0.050	0.046
16	0.052	0.057	0.058	0.055	0.056	0.055	0.057	0.063	0.058	0.057
32	0.054	0.075	0.079	0.080	0.084	0.088	0.086	0.100	0.084	0.081
64	0.093	0.145	0.165	0.181	0.194	0.198	0.202	0.215	0.198	0.202
128	0.158	0.307	0.389	0.427	0.463	0.469	0.499	0.504	0.500	0.490
256	0.185	0.435	0.603	0.690	0.742	0.773	0.775	0.796	0.782	0.791
512	0.144	0.410	0.633	0.793	0.872	0.898	0.924	0.930	0.928	0.935
cv1	0.240	0.192	0.170	0.151	0.143	0.144	0.140	0.141	0.139	0.137
cv2	1.335	1.434	1.493	1.517	1.509	1.524	1.500	1.472	1.510	1.526
$\kappa = \frac{1}{2}$										
$N =$ 8	0.054	0.051	0.051	0.048	0.052	0.058	0.049	0.054	0.054	0.053
16	0.055	0.060	0.061	0.055	0.055	0.058	0.064	0.062	0.066	0.069
32	0.074	0.079	0.086	0.085	0.086	0.096	0.089	0.086	0.098	0.090
64	0.107	0.166	0.181	0.193	0.205	0.213	0.219	0.216	0.222	0.218
128	0.167	0.312	0.387	0.425	0.443	0.471	0.464	0.473	0.482	0.491
256	0.188	0.409	0.544	0.625	0.666	0.690	0.709	0.712	0.715	0.730
512	0.136	0.345	0.539	0.667	0.743	0.798	0.819	0.833	0.848	0.857
cv1	0.049	0.050	0.047	0.043	0.043	0.041	0.040	0.041	0.041	0.040
cv2	0.874	0.826	0.817	0.817	0.814	0.792	0.793	0.791	0.779	0.773

Note: cv1 and cv2 are the 2.5% and 97.5% quantiles of the asymptotic distribution of $VR(K)$ under H_0, with $\kappa = K/T$, used to compute the power function. Hence, the power is for a two-sided test with nominal size 5%.

TABLE 2. Power of $VR(K)$ against H_2: $\gamma = -0.2$

T		8	16	32	64	128	256	512	1,024	2,048	∞
$\kappa = \frac{1}{16}$											
$N =$	8			0.048	0.043	0.045	0.056	0.047	0.051	0.045	0.044
	16			0.051	0.049	0.047	0.056	0.046	0.051	0.047	0.050
	32			0.053	0.054	0.063	0.065	0.060	0.069	0.060	0.061
	64			0.104	0.123	0.148	0.156	0.138	0.164	0.152	0.151
	128			0.263	0.377	0.471	0.523	0.509	0.551	0.526	0.531
	256			0.565	0.870	0.959	0.982	0.984	0.990	0.989	0.990
	512			0.777	0.997	1.000	1.000	1.000	1.000	1.000	1.000
cv1				0.609	0.524	0.496	0.482	0.464	0.472	0.462	0.460
cv2				1.270	1.388	1.442	1.442	1.462	1.484	1.488	1.473
$\kappa = \frac{1}{8}$											
$N =$	8		0.047	0.044	0.050	0.047	0.051	0.048	0.043	0.046	0.040
	16		0.044	0.050	0.045	0.044	0.050	0.048	0.043	0.045	0.045
	32		0.057	0.060	0.069	0.067	0.070	0.064	0.059	0.069	0.069
	64		0.105	0.153	0.182	0.176	0.204	0.206	0.175	0.204	0.193
	128		0.236	0.441	0.580	0.622	0.687	0.689	0.657	0.691	0.686
	256		0.358	0.816	0.973	0.993	0.998	0.998	0.999	0.999	0.999
	512		0.407	0.950	1.000	1.000	1.000	1.000	1.000	1.000	1.000
cv1			0.441	0.363	0.330	0.307	0.304	0.297	0.286	0.294	0.289
cv2			1.334	1.476	1.535	1.540	1.560	1.569	1.562	1.594	1.592
$\kappa = \frac{1}{4}$											
$N =$	8	0.050	0.043	0.045	0.051	0.041	0.044	0.049	0.046	0.043	0.048
	16	0.051	0.046	0.048	0.042	0.037	0.046	0.050	0.047	0.044	0.042
	32	0.062	0.057	0.061	0.068	0.061	0.072	0.077	0.076	0.066	0.069
	64	0.093	0.126	0.164	0.193	0.191	0.217	0.223	0.231	0.209	0.210
	128	0.133	0.274	0.465	0.603	0.647	0.714	0.737	0.738	0.724	0.735
	256	0.149	0.410	0.845	0.978	0.995	0.998	1.000	0.999	0.999	0.999
	512	0.152	0.445	0.953	1.000	1.000	1.000	1.000	1.000	1.000	1.000
cv1		0.248	0.193	0.164	0.150	0.140	0.142	0.142	0.140	0.136	0.138
cv2		1.330	1.475	1.498	1.476	1.559	1.501	1.484	1.514	1.515	1.499
$\kappa = \frac{1}{2}$											
$N =$	8	0.047	0.046	0.042	0.045	0.047	0.042	0.044	0.050	0.041	0.044
	16	0.045	0.040	0.040	0.045	0.043	0.039	0.040	0.042	0.038	0.046
	32	0.046	0.052	0.056	0.057	0.073	0.058	0.053	0.061	0.061	0.062
	64	0.064	0.087	0.118	0.139	0.177	0.157	0.158	0.163	0.158	0.185
	128	0.068	0.156	0.300	0.419	0.532	0.505	0.521	0.553	0.542	0.586
	256	0.073	0.221	0.608	0.862	0.950	0.961	0.968	0.974	0.975	0.983
	512	0.074	0.228	0.777	0.990	0.999	1.000	1.000	1.000	1.000	1.000
cv1		0.051	0.052	0.047	0.044	0.044	0.041	0.040	0.041	0.040	0.041
cv2		0.898	0.855	0.832	0.790	0.824	0.798	0.792	0.795	0.804	0.795

Note: cv1 and cv2 are the 2.5% and 97.5% quantiles of the asymptotic distribution of $VR(K)$ under H_0, with $\kappa = K/T$, used to compute the power function. Hence, the power is for a two-sided test with nominal size 5%.

TABLE 3. Power of $VR(K)$ against H_3: $\gamma = -0.2$, $\tau = \sigma^b/\sigma^a = \frac{1}{2}$

T	8	16	32	64	128	256	512	1,024	2,048	∞
$\kappa = \frac{1}{16}$										
$N =$ 8			0.048	0.049	0.051	0.054	0.050	0.044	0.053	0.062
16			0.052	0.046	0.044	0.051	0.050	0.047	0.050	0.056
32			0.057	0.050	0.056	0.055	0.056	0.054	0.057	0.059
64			0.078	0.085	0.095	0.106	0.108	0.096	0.099	0.113
128			0.155	0.214	0.244	0.286	0.293	0.284	0.297	0.306
256			0.264	0.475	0.604	0.697	0.734	0.730	0.747	0.773
512			0.267	0.609	0.850	0.937	0.967	0.974	0.977	0.987
cv1			0.613	0.528	0.492	0.480	0.472	0.460	0.462	0.465
cv2			1.274	1.374	1.431	1.455	1.475	1.482	1.463	1.466
$\kappa = \frac{1}{8}$										
$N =$ 8		0.045	0.051	0.045	0.043	0.049	0.045	0.044	0.044	0.049
16		0.044	0.046	0.044	0.047	0.045	0.051	0.045	0.043	0.043
32		0.051	0.051	0.050	0.057	0.056	0.057	0.057	0.055	0.051
64		0.074	0.095	0.103	0.115	0.116	0.116	0.120	0.114	0.120
128		0.115	0.181	0.248	0.289	0.314	0.330	0.321	0.302	0.327
256		0.114	0.271	0.427	0.560	0.618	0.666	0.659	0.657	0.679
512		0.085	0.211	0.436	0.665	0.778	0.834	0.845	0.850	0.866
cv1		0.437	0.362	0.323	0.310	0.299	0.297	0.294	0.287	0.292
cv2		1.329	1.464	1.528	1.582	1.562	1.575	1.611	1.583	1.602
$\kappa = \frac{1}{4}$										
$N =$ 8	0.046	0.045	0.045	0.048	0.053	0.048	0.043	0.045	0.044	0.041
16	0.044	0.041	0.043	0.046	0.049	0.044	0.041	0.039	0.040	0.039
32	0.048	0.050	0.051	0.054	0.063	0.061	0.053	0.055	0.054	0.053
64	0.057	0.068	0.088	0.108	0.125	0.125	0.111	0.109	0.113	0.108
128	0.064	0.103	0.157	0.212	0.266	0.269	0.256	0.261	0.257	0.255
256	0.062	0.087	0.179	0.294	0.412	0.442	0.449	0.456	0.468	0.460
512	0.051	0.067	0.134	0.275	0.451	0.541	0.549	0.590	0.592	0.595
cv1	0.237	0.191	0.164	0.152	0.150	0.144	0.138	0.137	0.137	0.136
cv2	1.336	1.462	1.440	1.504	1.507	1.521	1.549	1.509	1.522	1.548
$\kappa = \frac{1}{2}$										
$N =$ 8	0.048	0.044	0.048	0.044	0.044	0.041	0.046	0.044	0.047	0.045
16	0.049	0.045	0.045	0.042	0.046	0.041	0.046	0.041	0.043	0.036
32	0.047	0.049	0.045	0.050	0.050	0.045	0.048	0.049	0.049	0.047
64	0.055	0.060	0.069	0.072	0.083	0.082	0.086	0.090	0.092	0.074
128	0.050	0.061	0.106	0.122	0.152	0.161	0.170	0.179	0.180	0.158
256	0.047	0.059	0.119	0.183	0.243	0.268	0.289	0.304	0.313	0.289
512	0.048	0.054	0.092	0.180	0.275	0.339	0.379	0.403	0.421	0.383
cv1	0.053	0.052	0.048	0.043	0.042	0.040	0.040	0.040	0.040	0.038
cv2	0.889	0.845	0.819	0.808	0.798	0.793	0.806	0.802	0.820	0.795

Note: cv1 and cv2 are the 2.5% and 97.5% quantiles of the asymptotic distribution of $VR(K)$ under H_0, with $\kappa = K/T$, used to compute the power function. Hence, the power is for a two-sided test with nominal size 5%.

of the parameter γ are chosen to have a wide range of values for power given the configurations for N and T (i.e., not all close to one or the size of the test). Note that the power for alternative values of γ can be obtained from the tables. The relevant parameter is of the form $\exp(\gamma h)$, and a given diagonal corresponds to a particular value of $h = N/T$. Because N and T are doubled across each row or column the power for $\gamma^* = 2^i\gamma$ can be obtained by looking at the entries i diagonals below ($i > 0$) or i diagonals above ($i < 0$).

In general, for any given value of κ, we observe the following features: if N is small ($N \leq 16$), the power is close to the size; for a given fixed T, the power increases substantially with N; for a given N, the increase in power as T increases is important when T is small but becomes marginal when T reaches 128 or 256. Power depends much more on the span of the data than on the number of observations per se (see also Shiller and Perron, 1985; Perron, 1991). A feature of substantial interest is that, for any alternative considered, the power initially increases with an increase in κ and subsequently decreases as κ increases further. Results not shown indicate that, under H_1, an increase in $|\delta|$ reduces power but the asymptotic distribution remains a good approximation. Similarly, under H_3, an increase in τ decreases power.

Overall, the results also show that the asymptotic power functions are good approximations to the finite sample power unless the sample size is small. The approximations are better when κ is small but still adequate when κ is large.

4.1.1. Remarks on Alternative Approximations. As discussed in Section 3.4, the standard $K/T \to \kappa$ asymptotic framework used by Richardson and Stock (1989) and Deo and Richardson (2003) is not adequate to analyze power. Another popular approach to analyze the power function of tests is the so-called Bahadur's slope approximation. In the context of the variance ratio statistic, this was used by Richardson and Smith (1991). But this approach is tailored to assess relative power performance between two tests; hence it cannot provide a direct approximation to the power function for selected parameter configurations (it can, however, be used to select a candidate value of K that maximizes power; see Section 4.3). Our approach, on the other hand, can provide a direct approximation to the power function, and as we have shown it does provide a rather good approximation from which reliable relative rankings can be obtained. This, we believe, shows the usefulness of the asymptotic framework adopted.

4.2. Asymptotic Power as a Function of κ

Using simulations with 2,000 replications, we obtained the asymptotic power under H_1, H_2, and H_3 for $\kappa = 0.00 \ldots (0.02) \ldots 0.70$. For H_1, we set $c = \gamma N = -5$ and vary $g = \beta N$ from 0 to 28 (in steps of 2). We also consider $\delta = 0$, -0.5, and -0.9. For H_2, we vary $c = \gamma N$ from 0 to -28 (in steps of 2) whereas for H_3, we set $\tau = \frac{1}{2}$ and vary c from 0 to -80 (in steps of 4).

The results are presented in Figures 1–3 and Table 4 summarizes the importance of κ for the various experiments considered by presenting, for each case, the value of κ that maximizes asymptotic power. The results show, as expected, that the power is close to the size of the test when g is small, in which case variations in κ have no important effect. When g increases, variations in κ induce more important differences in power. When $\delta = 0$ or -0.5, the value of κ that yields maximal power is smaller the more distant g is from the null value. When $\delta = -0.9$, the value of κ that yields maximal power initially increases with an increase in g but eventually decreases with further increases in g. The results also show that when $c = -5$ (dividend-price ratio being locally stationary), the test can be biased for alternatives g close to 0 and that this bias increases as $|\delta|$ increases. In general, power decreases with an increase in $|\delta|$ when $c < 0$.

The alternatives H_2 and H_3 allow us to analyze the joint effect of c and κ on the power of the test. Given that, here, c measures the extent to which log-prices are far from being a random walk, Figure 2 shows that, for H_2, the faster prices revert to their mean value (c more negative), the stronger is the effect of variations in κ on power. Remarkably, a value of $\kappa = 0.22$ yields maximal power for all values of c considered. Under the alternative H_3, the power is again substantially affected by κ. As for H_1, the value of κ that yields maximal power decreases as the alternative gets further away from the null.

To summarize, the asymptotic distributions obtained with $K/T \to \kappa$ provide adequate approximations to the finite sample distributions under both the null hypothesis and the three alternatives selected. A salient feature of the power function under all these alternatives is that power initially increases and then decreases as κ increases, which accords with the simulation results of Lo and Mackinlay (1989).

4.3. Comparison with Bahadur's Approximate Slope Function

An alternative approach that delivers a prediction about the value of K that maximizes power is the approximate slope function developed by Bahadur (1960) and extended by Geweke (1981). When the limit distribution of a statistic is chi square under the null hypothesis (as is the square of the variance ratio statistic under the standard asymptotic framework), then the approximate slope equals the probability limit of the statistic under the alternative hypothesis deflated by the sample size. Among asymptotically valid statistics, the one with a maximal approximate slope is predicted to have better power. Richardson and Smith (1991) used this approach to analyze the value of K that would maximize the power of the variance ratio statistic in the context of an AR(1) alternative as specified by our alternative H_2. In this section, we revisit the analysis and results underlying their Table 7. They showed that when the alternative is an autoregressive parameter 0.95 ($= \exp(\gamma h)$ in our notation), Bahadur's approximation suggests that $K = 48$ will maximize power, this value being the same

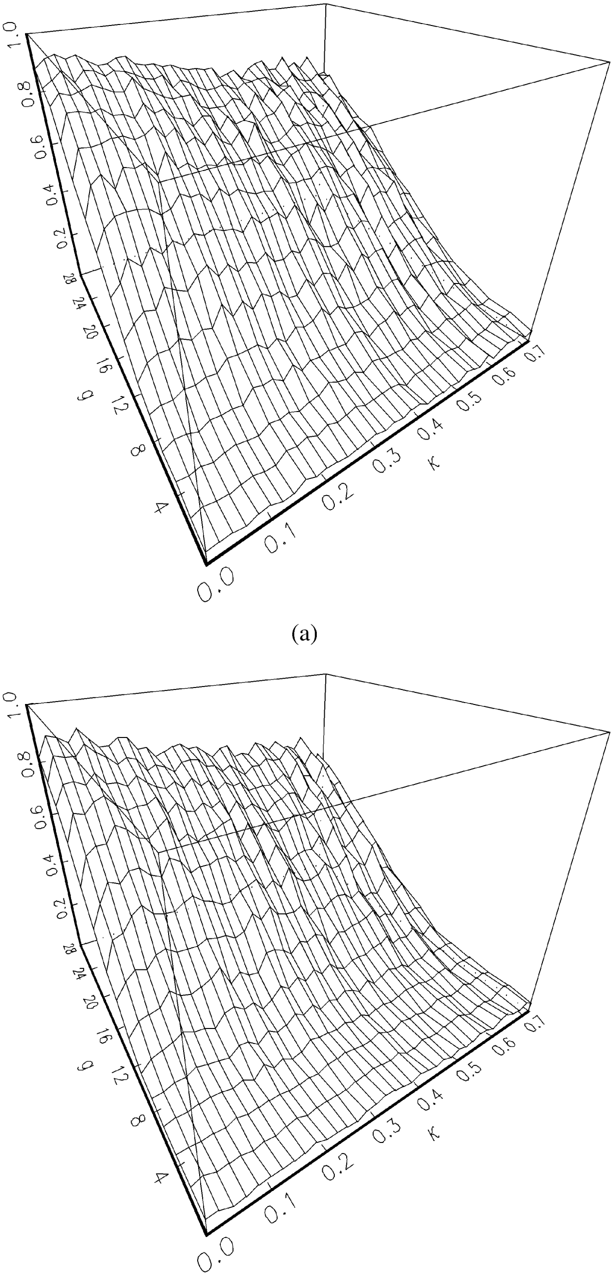
Table 4. Values of κ that maximize power for selected alternatives

	H_1	H_1	H_1			H_3
$-c\backslash g$	$\delta = 0$	$\delta = -0.5$	$\delta = -0.9$	H_2	c	$\tau = 0.5$
2	0.60	0.72	0.34	0.22	4	0.52
4	0.30	0.42	0.34	0.22	8	0.24
6	0.50	0.20	0.34	0.22	12	0.16
8	0.36	0.30	0.34	0.22	16	0.24
10	0.18	0.38	0.34	0.22	20	0.16
12	0.18	0.20	0.46	0.22	24	0.16
14	0.10	0.34	0.46	0.22	28	0.16
16	0.18	0.10	0.46	0.22	32	0.08
18	0.18	0.18	0.18	0.22	36	0.08
20	0.18	0.10	0.16	0.22	40	0.08
22	0.04	0.10	0.12	0.22	44	0.08
24	0.18	0.10	0.12	0.22	48	0.08
26	0.06	0.10	0.10	0.22	52	0.08
28	0.06	0.10	0.04	0.22	56	0.08
					60	0.04
					64	0.04
					68	0.04
					72	0.04
					76	0.08
					80	0.04

for all sample sizes. In contrast, our approach suggests that a value of $K = 0.22T$ will maximize power.[2]

We replicated Richardson and Smith's (1991) experiment with $T = 720$ (as they did) and $T = 360$ also. For reasons that will become clear, we evaluate power using (1) the standard normal critical values suggested by the usual fixed K asymptotic framework and (2) the critical values from the asymptotic framework in which $K/T \to \kappa$. The results are presented in Table 5. As can be seen from the first column, when $T = 720$ the approximate slope does well in selecting the value of K that maximizes power when the fixed K asymptotic critical values are used.[3] Because these imply tests with large size distortions when K is large (too conservative), it can be said to perform well with non-size-adjusted power. However, from the results in the second column, it is seen to be very sensitive to the sample size used. Changing the sample size to $T = 360$, a value $K = 48$ yields very low power (compared to the highest possible at $K = 24$).

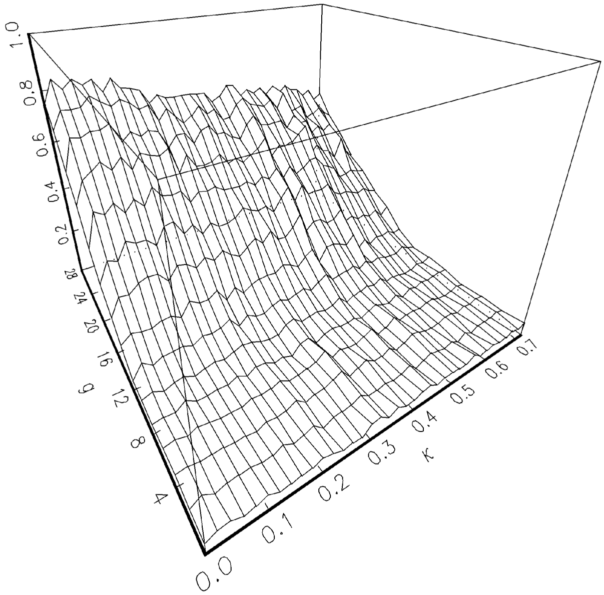
Things are very different when evaluating power with critical values from the $K/T \to \kappa$ asymptotic framework, which is basically equivalent to analyzing size-adjusted power because the tests then have very little size distortion. When $T = 720$, our approach suggests that $K = 158$ will maximize power, which is



(a)

(b)

FIGURE 1. Asymptotic power under H_1: (a) $c = -5$, $\delta = 0$; (b) $c = -5$, $\delta = -0.5$; (c) $c = -5$, $\delta = -0.9$. (*Figure continues on next page.*)



(c)

Figure 1. *Continued.*

indeed the case. The power is 0.97 compared to 0.85 when $K = 48$ is selected. More important, our approach is robust to changes in the sample size. When $T = 360$, our framework predicts that $K = 79$ will maximize power, which is close to the maximum attainable.

From these simulations, we can conclude that our approach is better suited than the slope approximation to provide a value of K that will maximize power, provided the $K/T \to \kappa$ asymptotic critical values are used, which should be done in any event to avoid highly size-distorted tests.

4.4. The Effect of Nonzero Initial Conditions

We now consider the effect of a nonzero initial value on the power of the test under the different alternatives (the size is unaffected because the statistic is invariant to the initial value under the null hypothesis). To illustrate the qualitative results, we consider the alternatives H_1 with $c = -5$ and $\delta = 0$, and also H_2 and H_3 with $\tau = \frac{1}{2}$. We simulated the power function of the variance ratio statistic for values of the normalized initial conditions between 0 and 6. The results are presented in Figure 4.

The results show that, in general, a nonzero initial value increases the power of the test. This increase is bigger the further away prices are from being a

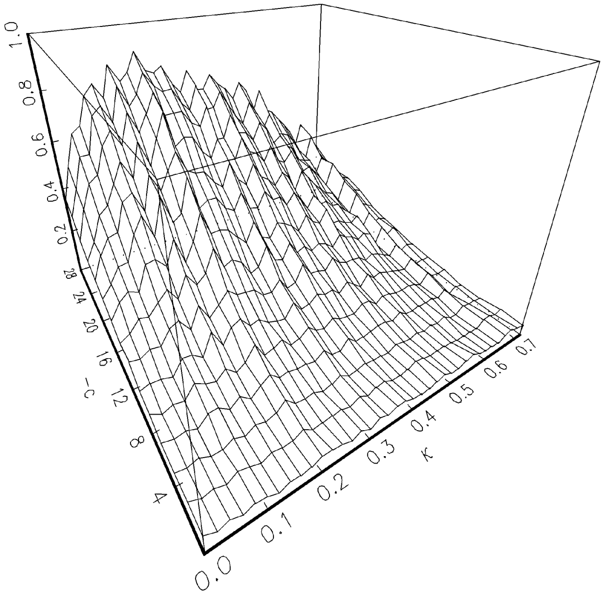


FIGURE 2. Asymptotic power under H_2: $\tau = 0$.

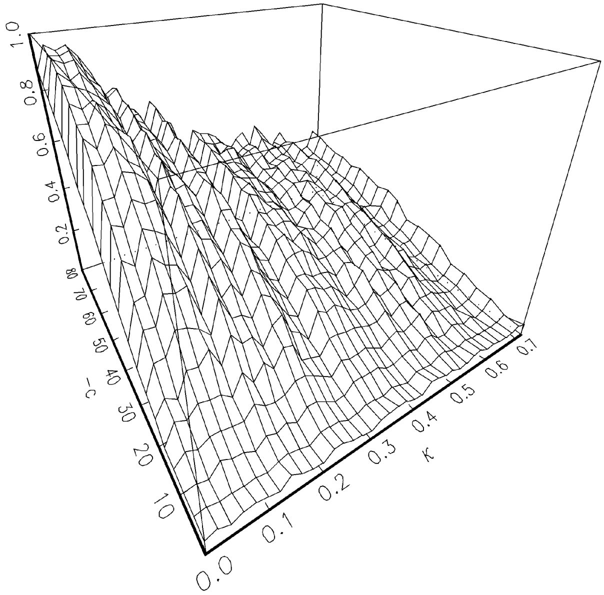


FIGURE 3. Asymptotic power under H_3: $\tau = \frac{1}{2}$.

Table 5. Finite sample power, alternative H_2, $\exp(\gamma h) = 0.95$

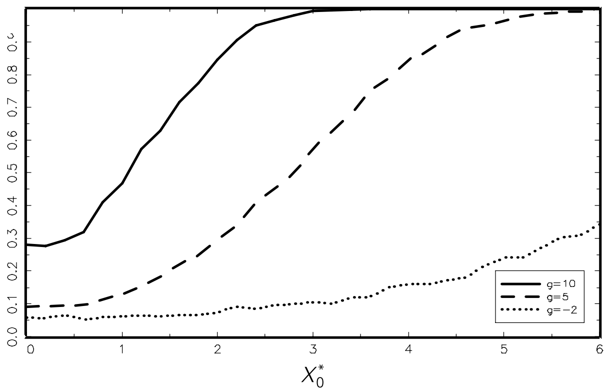
	Power using fixed K asymptotic critical values		Power using $K/T \to \kappa$ asymptotic critical values	
K	$T = 720$	$T = 360$	$T = 720$	$T = 360$
12	0.42	0.19	0.41	0.19
24	0.64	0.23	0.66	0.29
36	0.75	0.19	0.79	0.35
48	0.80	0.08	0.85	0.38
60	0.82	0.00	0.90	0.41
72	0.79	0.00	0.92	0.45
84	0.72	0.00	0.95	0.43
96	0.57	0.00	0.95	0.40
108	0.30	0.00	0.96	0.41
120	0.04	0.00	0.95	0.39
132	0.00	0.00	0.96	0.39
144	0.00	0.00	0.96	0.37
156	0.00	0.00	0.97	0.36
168	0.00	0.00	0.97	0.33
180	0.00	0.00	0.97	0.32
192	0.00	0.00	0.96	0.29
204	0.00	0.00	0.96	0.29
216	0.00	0.00	0.95	0.28
228	0.00	0.00	0.95	0.24
240	0.00	0.00	0.94	0.22

Note: The value of K predicted to maximize power by the Bahadur slope approach is 48 for both $T = 720$ and $T = 360$. The value of K predicted to maximize power using our continuous-time approach is $K = 158$ for $T = 720$ and $K = 79$ for $T = 360$.

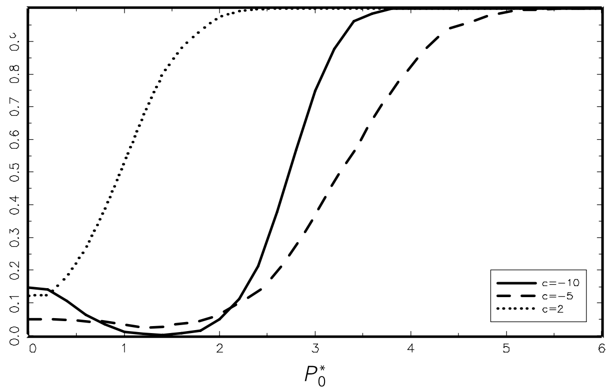
random walk. This result is fairly intuitive. When prices have a stationary component, the larger the initial value the further away from the unconditional mean is the process at time 0. Hence, the movement at the beginning shows a strong reversion to the mean, the reversion appearing stronger the larger the initial value. When prices have an explosive component, the effect of a nonzero initial value is to exacerbate the speed at which the process departs from its initial value. Hence, it becomes even easier to distinguish from a random walk.

There are some exceptions that show that the power can initially decrease with an increase in the initial value. This occurs, most notably, when prices are strongly mean-reverting (alternative H_2 with a large negative c). This is due to the fact that a nonzero initial value has the effect of increasing the variance ratio whether the process is mean-reverting or explosive. With a mean-reverting process, the variance ratio take values below one, and we reject for small values. Hence, this increase causes a decrease in power. Eventually, as the initial value gets larger the mean reversion effect dominates.

(a) H_1: $c = -5$, $\delta = 0$



(b) H_2: $\tau = 0$



(c) H_3: $\tau = 1/2$

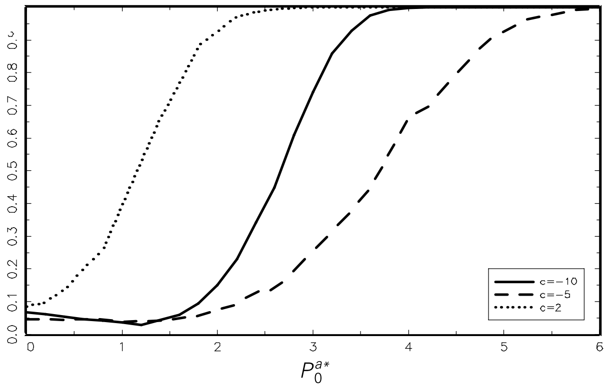


FIGURE 4. Effect of a nonzero initial value on asymptotic power.

5. A TEST THAT DOES NOT DEPEND ON κ

The fact that the power function of the variance ratio statistic for alternatives of interest attains a maximum for some value of κ means that power losses can be important if κ is not chosen appropriately. The value of κ that maximizes power depends, however, on the underlying true data-generating process under the alternative hypothesis, which is unknown. Hence, a useful strategy is to look at the variance ratio statistic over a range of values for κ. To motivate the approach adopted consider first the case where the alternative is right-sided, i.e., we reject for large values of the variance ratio statistic. A common strategy is to find the value of κ that leads to the strongest possible rejection. This leads to the test statistic $VR_{\max} = \sup_{\kappa\in[\varepsilon,1-\varepsilon]} VR([T\kappa])$, which looks for the maximal value of the variance ratio statistic. Here, ε is some trimming parameter that defines the range of permissible values for κ. Applying the same logic to the case of left-sided alternatives (rejecting for low values of the statistics) would lead us to consider the minimal value of the variance ratio statistic over some range for κ. This approach, however, leads to a test with no power because the distribution of $VR(K)$, both in finite samples and asymptotically, has some mass at 0. Hence, the distribution of the minimal value has a rather important mass at 0, and the critical values (using standard significance levels) are basically 0. Because the statistic is bounded below by 0, this implies basically no power. For this reason, we continue to consider the $VR_{\max}$ statistic even for left-sided alternatives. The interpretation is then that one is looking at the least favorable case for the alternative and whether it still implies a rejection. Hence, for left-sided alternatives, it should be viewed as a conservative procedure.[4]

Using the result of Proposition 1, the asymptotic distribution of $VR_{\max}$ is

$$VR_{\max} \Rightarrow \sup_{\kappa\in[\varepsilon,1-\varepsilon]} \left(\frac{1}{\kappa} \int_{\kappa}^{1} [W(r) - W(r-\kappa) - \kappa W(1)]^2 \, dr \right).$$

Table 6 presents the quantiles of the asymptotic distribution and also those of the finite sample distributions for selected values of T and ε. Two features of interest are noteworthy from these results. First, the asymptotic distribution is, in general, a good approximation to the finite sample distributions. Second, changing ε has little effect on the quantiles of the distribution in the right tail but substantial effects in the left tail. This suggests that different values of ε are unlikely to affect the power of the test when the alternative is such that the variance ratio increases above 1. However, the choice of ε could substantially affect power when the alternative is such that the variance ratio decreases below 1.

To assess the extent of the potential power losses compared to that attainable with the "best" κ, we simulated the asymptotic power function of the test $VR_{\max}$ when the alternative is either H_1 or H_2. Consider first the asymptotic power function when the alternative is H_1 with $c = \delta = 0$ (Figure 5). Here returns are influenced by the dividend-price ratio, which is modeled as a nearly integrated

Table 6. Quantiles of the distribution of $\max_{\kappa\in[\varepsilon,1-\varepsilon]} VR([T\kappa])$ under H_0

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
$\varepsilon = 0.01$									
$T = 80$	1.000	1.000	1.000	1.000	1.012	1.341	1.489	1.621	1.840
120	0.787	0.824	0.850	0.890	1.029	1.332	1.510	1.709	1.864
180	0.832	0.855	0.884	0.909	1.035	1.337	1.486	1.711	2.003
360	0.764	0.802	0.835	0.876	1.029	1.344	1.514	1.715	2.004
720	0.754	0.779	0.823	0.857	1.028	1.380	1.552	1.713	2.039
∞	0.746	0.784	0.812	0.854	1.034	1.388	1.570	1.697	1.949
$\varepsilon = 0.05$									
$T = 80$	0.525	0.580	0.625	0.693	0.948	1.332	1.504	1.705	1.935
120	0.522	0.564	0.604	0.667	0.940	1.354	1.545	1.751	1.946
180	0.494	0.538	0.588	0.652	0.920	1.358	1.551	1.735	2.049
360	0.462	0.496	0.561	0.621	0.914	1.328	1.480	1.683	1.872
720	0.461	0.522	0.571	0.630	0.900	1.350	1.514	1.690	1.898
∞	0.457	0.522	0.572	0.631	0.906	1.339	1.551	1.765	2.020
$\varepsilon = 0.10$									
$T = 80$	0.356	0.394	0.452	0.511	0.817	1.308	1.480	1.619	2.035
120	0.319	0.377	0.421	0.474	0.787	1.247	1.462	1.605	1.864
180	0.309	0.359	0.418	0.493	0.785	1.282	1.456	1.646	1.909
360	0.309	0.350	0.401	0.471	0.777	1.314	1.548	1.723	1.972
720	0.313	0.361	0.405	0.457	0.768	1.335	1.506	1.696	2.055
∞	0.289	0.331	0.385	0.456	0.759	1.267	1.445	1.687	1.950
$\varepsilon = 0.15$									
$T = 80$	0.234	0.275	0.316	0.382	0.694	1.224	1.424	1.642	1.900
120	0.231	0.270	0.302	0.355	0.670	1.234	1.422	1.637	1.904
180	0.209	0.253	0.294	0.356	0.665	1.253	1.430	1.623	1.825
360	0.218	0.262	0.305	0.360	0.679	1.210	1.447	1.642	1.965
720	0.211	0.251	0.288	0.348	0.668	1.235	1.441	1.627	1.824
∞	0.207	0.247	0.294	0.344	0.651	1.240	1.465	1.659	1.955
$\varepsilon = 0.20$									
$T = 80$	0.173	0.204	0.246	0.291	0.581	1.135	1.352	1.568	1.794
120	0.168	0.202	0.226	0.282	0.580	1.127	1.340	1.580	1.940
180	0.159	0.191	0.233	0.285	0.578	1.140	1.336	1.587	1.858
360	0.160	0.187	0.221	0.275	0.588	1.168	1.392	1.626	1.926
720	0.145	0.188	0.226	0.281	0.570	1.163	1.373	1.649	1.983
∞	0.144	0.173	0.211	0.271	0.566	1.150	1.384	1.650	1.912

process. This is a case in which the variance ratio statistic takes values above 1 (increasing as the sample size increases). As expected, the power function is basically invariant to the value of the trimming used. It is somewhat below that attainable with the best fixed κ but substantially above that obtained with other

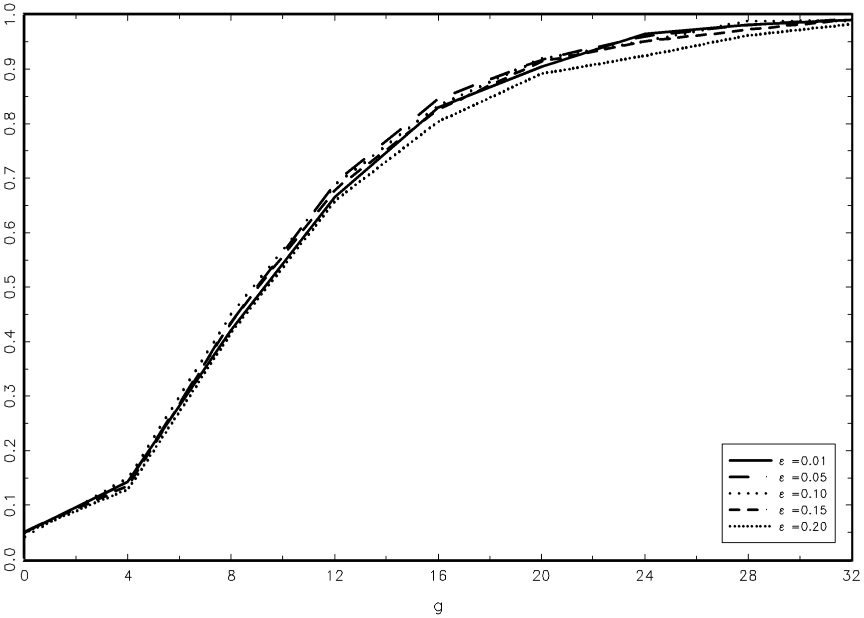


FIGURE 5. Asymptotic power function of $VR_{\max}$: alternative H_1.

possible choices for κ. Hence, the test provides a useful way to deal with the fact that the best κ is unknown in practice while retaining reasonable power.

Consider now the case when the alternative is H_2 (Figure 6). Here prices are stationary, which causes the variance ratio to take values below 1 (approaching 0 as T increases). The power function is now severely affected by the choice of the trimming value ε. Indeed, power decreases as ε approaches 0. This is due to the fact that when $K = 0$, $VR(K) = 1$ by definition. Hence, for small values of the trimming the maximal value of $VR(K)$ has to be, basically by construction, close to 1; hence, the associated power loss.

Because the trimming is inconsequential for the right-sided alternative and a moderate to large trimming is preferable for the left-sided alternative, a sensible strategy is to use a trimming in the range 0.10 to 0.20.

6. CONCLUSION

We considered a test of market efficiency based on the variance ratio statistic. Our framework was to posit a continuous-time process of interest, derive its discrete-time representation, and then take the limit as the sample size T increases, keeping the span of the data fixed (i.e., letting the sampling interval tend to zero at rate T). We described the limit distribution of the variance ratio

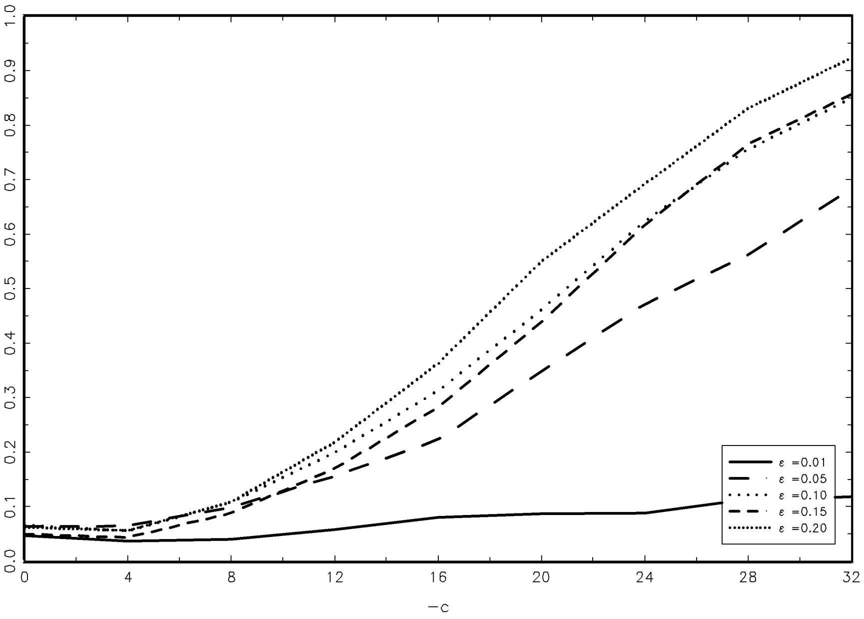


FIGURE 6. Asymptotic power function of $VR_{\max}$: alternative H_2.

statistic under the null hypothesis of market efficiency defined by uncorrelated returns and under three alternative hypotheses of interest: namely, with the dividend-price ratio as a predictor of returns, with trend-stationary prices, and with prices as the sum of a permanent and transitory component. Simulations have shown that the limiting distributions obtained are good approximations to the finite sample distributions. An analysis of the power functions showed that, under all alternatives considered, an increase in the aggregation parameter κ induces an initial increase in power followed by a decrease as κ is increased further. Hence, for a given alternative hypothesis there is a value of K relative to T that maximizes power. This is contrary to regression-based tests, e.g., regressing K-period returns on the dividend-price ratio (see, e.g., Fama and French, 1988b), in which case Perron and Vodounou (2004) showed that power decreases monotonically as κ increases. For that reason, we have also considered the $VR_{\max}$ test, which looks at the maximal value of the variance ratio statistic over a prespecified range for K. We have shown that this leads to a test with power close to that attainable with the "best" value of κ. We have also shown that care must be exercised in choosing the trimming that defines the range of values for K considered and that a very small trimming should be avoided.

NOTES

1. We also performed the experiments with $\delta = -0.5$ and -0.9 but do not report the results explicitly though we shall comment on the differences.

2. An autoregressive parameter 0.95 with $T = 720$ implies a value of $\gamma = -37$, approximately. Although this value is outside the range of Table 4, we have verified that the value of κ that maximizes power is still 0.22.

3. The power entries in the first column of Table 5 are higher than those reported in Table 7 of Richardson and Smith (1991) for reasons unknown to us. Nevertheless, the shape of the power function is generally the same, and the same conclusions follow.

4. As suggested by a referee, it would be possible to use a cross-validation procedure to select the best value of κ. Such an extension is not reported here.

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APPENDIX: PROOF OF PROPOSITIONS 1 AND 2

The statistic $VR(K)$ being invariant to linear transformation of R_{th}, we use the normalized process $Z_{th}^* = Z_{th}/h^{1/2}$. Throughout, we make extensive use of the following results, which are now standard.

LEMMA A.1. *Let* $u_{th} = (\epsilon_{th}^*, v_{th}^*)' = (\epsilon_{th}, v_{th})'/h^{1/2}$ *be i.i.d.* $(0, \Sigma)$ *with* Σ *defined by (7). Then, (i)* $T^{-1/2}\sum_{t=1}^{[Tr]}\epsilon_{th}^* \Rightarrow \sqrt{1-\delta^2}W_1(r) + \delta W_2(r)$; *(ii)* $T^{-1/2}\sum_{t=1}^{[Tr]} v_{th}^* \Rightarrow \sqrt{\tau}W_2(r)$ *(jointly) where* $W_1(r)$ *and* $W_2(r)$ *are independent Wiener processes defined on* $C[0,1]$ *and* $\Rightarrow$ *denotes weak convergence in distribution. Also, (iii)* $T^{-1}\sum_{t=1}^{T}\epsilon_{th}^{*2} \to_p 1$.

To obtain the limit distribution of $VR(K)$ under H_1, we first consider the following lemma related to sample moments.

LEMMA A.2. *Let* (R_{th}, X_{th}, e_{th}) *be generated by (10) and* $(R_{th}^*, X_{th}^*, e_{th}^*) = (R_{th}, X_{th}, e_{th})/h^{1/2}$. *Also, let* $g = \beta N$ *and* $T = N/h$; *then as* $T \to \infty$ *with* N *fixed, we have, with* $J_2(r) = \int_0^r \exp(c(r-s))\,dW_2(s)$ *and* $W_{12}(r) = \sqrt{1-\delta^2}W_1(r) + \delta W_2(r)$,

(i) $T^{-1/2}X_{([Tr]-[T\kappa]+1)h}^* \Rightarrow \sqrt{\tau}J_2(r-\kappa) + \exp(c(r-\kappa))X_0^*$;
(ii) $T^{-3/2}\sum_{i=0}^{[T\kappa]-1}X_{([Tr]-1-i)h}^* \Rightarrow \int_{r-\kappa}^{r}[\sqrt{\tau}J_2(s) + \exp(cs)X_0^*]\,ds$;
(iii) $T^{-1/2}R_{[Tr]}^*([T\kappa]) \Rightarrow g[\int_{r-\kappa}^{r}[\sqrt{\tau}J_2(s) + \exp(cs)X_0^*]\,ds + W_{12}(r) - W_{12}(r-\kappa)]$.

Proof. Using Lemma A.1, we have, for part (i),

$$\begin{aligned}
T^{-1/2}X^*_{([Tr]-[T\kappa]+1)h} &= T^{-1/2}\sum_{j=1}^{[Tr]-[T\kappa]+1}\exp(\gamma h([Tr]-[T\kappa]+1-j))v^*_{jh}\\
&\quad + T^{-1/2}\exp(\gamma h([Tr]-[T\kappa]+1))X_0/h^{1/2}\\
&= \sum_{j=1}^{[Tr]-[T\kappa]+1}\exp(\gamma N([Tr]-[T\kappa]+1-j)/T)\int_{(j-1)/T}^{j/T} dX_{2T}(s)\\
&\quad + \exp(\gamma N([Tr]-[T\kappa]+1)/T)X_0/N^{1/2}\\
&= \sum_{j=1}^{[Tr]-[T\kappa]+1}\int_{(j-1)/T}^{j/T}\exp(c([Tr]-[T\kappa]+1-[Ts])/T)\,dX_{2T}(s)\\
&\quad + \exp(c([Tr]-[T\kappa]+1)/T)X_0^*\\
&\Rightarrow \sqrt{\tau}J_2(r-\kappa)+\exp(c(r-\kappa))X_0^*.
\end{aligned}$$

For part (ii)

$$\begin{aligned}
T^{-3/2}\sum_{i=0}^{[T\kappa]-1}X^*_{([Tr]-1-i)h} &= T^{-1}\sum_{j=[Tr]-[T\kappa]+1}^{[Tr]}(T^{-1/2}X^*_{(j-1)h})\\
&\Rightarrow \int_{r-\kappa}^{r}[\sqrt{\tau}J_2(s)+\exp(cs)X_0^*]\,ds.
\end{aligned}$$

For part (iii)

$$\begin{aligned}
T^{-1/2}R^*_{[Tr]h}([T\kappa]) &= N\frac{\alpha_h}{h}T^{-1}\sum_{i=0}^{[T\kappa]-1}(T^{-1/2}X^*_{([Tr]-1-i)h}) + T^{-1/2}\sum_{i=0}^{[T\kappa]-1}\epsilon^*_{([Tr]-i)h}\\
&\Rightarrow g\int_{r-\kappa}^{r}[\sqrt{\tau}J_2(s)+\exp(cs)X_0^*]\,ds + W_{12}(r)-W_{12}(r-\kappa),
\end{aligned}$$

because $(a_h/h)\to\beta$. The variance ratio statistic can be written as

$$VR(K) = K^{-1}T\,\frac{T^{-1}\sum_{t=K-1}^{T}(T^{-1/2}R^*_{th}(K)-T^{-1/2}K\hat{\mu}^*)^2}{T^{-1}\sum_{t=1}^{T}(R^*_{th}-\hat{\mu}^*)^2}$$

with $\hat{\mu}^* = T^{-1}\sum_{t=1}^{T} R^*_{th}$. For the denominator, we have

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}(R^*_{th}-\hat{\mu}^*)^2 &= T^{-1}\sum_{t=1}^{T}(R^*_{th})^2-\hat{\mu}^{*2}\\
&= T^{-1}\sum_{t=1}^{T}(\alpha_h X^*_{(t-1)h}+\epsilon^*_{th})^2-\hat{\mu}^{*2}\\
&= (\alpha_h^2/h^2)N^2T^{-3}\sum_{t=1}^{T}X^{*2}_{(t-1)h}\\
&\quad + 2(\alpha_h/h)NT^{-2}\sum_{t=1}^{T}X^*_{(t-1)h}\epsilon^*_{th}+T^{-1}\sum_{t=1}^{T}\epsilon^{*2}_{th}\\
&\Rightarrow 1,
\end{aligned}$$

using Lemma A.1(iii), because $\hat{\mu}^* \to_p 0$ (see the discussion that follows) and $\sum_{t=1}^{T} X^{*2}_{(t-1)h} = O_p(T^2)$, $\sum_{t=1}^{T} X^*_{(t-1)h}\epsilon^*_{th} = O_p(T)$. For the numerator, we have

$$\begin{aligned}
&T^{-1}\sum_{t=K-1}^{T}(T^{-1/2}R^*_{th}(K)-T^{-1/2}K\hat{\mu}^*)^2\\
&\quad\Rightarrow \int_{\kappa}^{1}\left[g\sqrt{\tau}\int_{r-\kappa}^{r}J_2(s)\,ds+W_{12}(r)-W_{12}(r-\kappa)\right.\\
&\qquad\qquad \left. -\,\kappa\left[g\int_0^1[\sqrt{\tau}J_2(s)+\exp(cs)X_0^*]\,ds+W_{12}(1)\right]\right]^2 dr
\end{aligned}$$

because

$$\begin{aligned}
T^{-1/2}K\hat{\mu}^* &= (K/T)T^{-1/2}\sum_{t=1}^{T}R^*_{th}\\
&= (K/T)T^{-1/2}\sum_{t=1}^{T}(\alpha_h X^*_{(t-1)h}+\epsilon^*_{th})\\
&= (K/T)\left[(\alpha_h/h)NT^{-3/2}\sum_{t=1}^{T}X^*_{(t-1)h}+T^{-1/2}\sum_{t=1}^{T}\epsilon^*_{th}\right]\\
&\Rightarrow \kappa\left[g\int_0^1[\sqrt{\tau}J_2(s)+\exp(cs)X_0^*]\,ds+W_{12}(1)\right].
\end{aligned}$$

The result of Proposition 2 follows. The limit distribution under the null hypothesis (Proposition 1) is obtained setting $\beta = 0$ (hence, $g = 0$) and noting that the marginal distribution of $W_{12}(r)$ is that of a standard Wiener process. ■

Proof of (13). We use the normalized process

$$P^*_{th} = \exp(\gamma h)P^*_{(t-1)h} + \varepsilon^*_{th},$$

where $P^*_{th} = P_{th}/a(h)^{1/2}$ and $\varepsilon^*_{th} = \varepsilon_{th}/a(h)^{1/2}$. Because $T^{-1}\sum_1^T(R^*_{th} - \hat{\mu}^*)^2 \Rightarrow 1$, we only consider the limit of the numerator. We first have

$$\begin{aligned}
T^{-1/2}P^*_{[Tr]h} &= T^{-1/2}\sum_{j=1}^{[Tr]}\exp(\gamma h([Tr]-j))\varepsilon^*_{th} + T^{-1/2}\exp(\gamma h[Tr])P^*_0 \\
&= \sum_{j=1}^{[Tr]}\exp\left(\gamma N\frac{[Tr]-j}{T}\right)\int_{(j-1)/T}^{j/T} dX_T(s) + T^{-1/2}\exp\left(\gamma N\frac{[Tr]}{T}\right)\frac{P_0}{a(h)^{1/2}} \\
&= \int_0^r \exp\left(c\,\frac{[Tr]-[Ts]}{T}\right)dX_T(s) + \exp\left(c\,\frac{[Tr]}{T}\right)\frac{N^{-1/2}P_0}{(a(h)/h)^{1/2}} \\
&\Rightarrow J_c(r) + \exp(cr)P^*_0
\end{aligned}$$

using $c = \gamma N$ and $a(h)/h \to \sigma^2$. From this result we deduce that

$$\begin{aligned}
T^{-1/2}R^*_{[Tr]h}([Tk]) &= T^{-1/2}\left[\exp(\gamma h)\sum_{i=0}^{[Tk]-1}P^*_{([Tr]-1-i)h} + \sum_{i=0}^{[Tk]-1}\varepsilon^*_{([Tr]-i)h}\right] \\
&= \left(\frac{\exp(\gamma h)}{h}\right)NT^{-1}\sum_{i=0}^{[Tk]-1}(T^{-1/2}P^*_{([Tr]-1-i)h}) \\
&\quad + T^{-1/2}\sum_{i=0}^{[Tk]-1}\varepsilon^*_{([Tr]-i)h} \\
&\Rightarrow c\int_{r-k}^{r}[J_c(s) + \exp(cs)P^*_0]\,ds + W(r) - W(r-k)
\end{aligned}$$

and

$$\begin{aligned}
T^{-1/2}K\hat{\mu}^* &= \frac{K}{T}T^{-1/2}\sum_{t=1}^{T}R^*_{th} \\
&= \frac{K}{T}T^{-1/2}\sum_{t=1}^{T}(\exp(\gamma h)P^*_{(t-1)h} + \varepsilon^*_{th}) \\
&= \frac{K}{T}\left[\frac{\exp(\gamma h)}{h}NT^{-1}\sum_{t=1}^{T}(T^{-1/2}P^*_{(t-1)h}) + T^{-1/2}\sum_{t=1}^{T}\varepsilon^*_{th}\right] \\
&\Rightarrow \kappa\left[c\int_0^1[J_c(s) + \exp(cs)P^*_0]\,ds + W(1)\right],
\end{aligned}$$

from which equation (13) follows. ■

Proof of (15). We start by noting that

$$\begin{aligned}
R_{th} &= P_{th} - P_{(t-1)h} = P^a_{th} - P^a_{(t-1)h} + P^b_{th} - P^b_{(t-1)h} \\
&= R^a_{th} + R^b_{th}.
\end{aligned}$$

Hence, returns are the sum of the “permanent” and “transitory” returns. We also define

$$\hat{\mu} = T^{-1}\sum_{t=1}^{T} R_{th} = T^{-1}\sum_{t=1}^{T} R_{th}^{a} + T^{-1}\sum_{t=1}^{T} R_{th}^{b} \equiv \hat{\mu}_a + \hat{\mu}_b.$$

Using the preceding notation, we can express the numerator, normalized by $h^{1/2}$, as

$$\frac{1}{K/T}\int_{\kappa}^{1} (T^{-1/2}R^{*}_{[Tr]h}([T\kappa])^{a} - T^{-1/2}\hat{\mu}^{*}_{a} + T^{-1/2}R^{*}_{[Tr]h}([T\kappa])^{b} - T^{-1/2}\hat{\mu}^{*}_{b})^2\, dr.$$

Now, $R^{*}_{[Tr]h}([T\kappa])^{a}$ are returns corresponding to stationary mean-reverting prices; hence we can apply the results corresponding to H_2. Similarly, $R^{*}_{[Tr]h}([T\kappa])^{b}$ are returns corresponding to random walk prices; hence we can apply the results corresponding to H_0. We then have the following limit:

$$\frac{(\sigma^{a})^2}{\kappa}\int_{\kappa}^{1}\left[\begin{array}{c} c\displaystyle\int_{r-\kappa}^{r}[J_c^{a}(s) + \exp(cs)P_0^{a*}]\,ds + W_a(r) - W_a(r-\kappa) \\ -\kappa\left[c\displaystyle\int_{0}^{1}[J_c^{a}(s) + \exp(cs)P_0^{a*}]\,ds + W_a(1)\right] \\ + (\sigma_b/\sigma_a)\{W_b(r) - W_b(r-\kappa) - \kappa W_b(1)\} \end{array}\right]^2 dr$$

$$= \frac{(\sigma^{a})^2}{\kappa}\int_{\kappa}^{1}\left[c\int_{r-\kappa}^{r}[J_c^{a}(s) + \exp(cs)P_0^{a*}]\,ds + W(r) - W(r-\kappa) - \kappa\left(c\int_{0}^{1}[J_c^{a}(s) + \exp(cs)P_0^{a*}]\,ds + W(1)\right)\right]^2 dr,$$

where $W(r) = W_b(r) + \tau W_a(r)$ with $\tau = \sigma_b/\sigma_a$. Next, we check the denominator.

$$T^{-1}\sum_{1}^{T}(R^{*}_{th} - \hat{\mu}^{*})^2 = T^{-1}\sum_{1}^{T}(R^{a*}_{th})^2 + T^{-1}\sum_{1}^{T}(R^{b*}_{th})^2 + o_p(1)$$

$$\to_p (\sigma^{a})^2 + (\sigma^{b})^2 = (\sigma^{a})^2(1+\tau^2).$$

The proof of (15) follows. ■