Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Empirical Finance 11 (2004) 203–230
Journal of EMPIRICAL FINANCE
www.elsevier.com/locate/econbase

Tests of return predictability: an analysis of their properties based on a continuous time asymptotic framework

Pierre Perron[a,*], Cosme Vodounou[b]

[a] *Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, USA*
[b] *Institut National de la Statistique et de l'Analyse Économique, Cotonou, Bénin*

Accepted 15 April 2003

Abstract

We consider the statistical properties of tests for return predictability based on regressing returns or multi-period returns on some variable such as the dividend/price ratio. We use a continuous time asymptotic framework whereby we let the sample size increase to infinity keeping the span of the data fixed. The data generating process specifies that prices and dividends are a multivariate Ohrnstein–Uhlenbeck process which encompasses the null and alternative hypotheses that returns are uncorrelated or correlated with the dividend/price ratio. For the multi-period returns case, say K-periods, we let $K/T \to \kappa$ as in [Journal of Financial Economics 25 (1989) 323]. We derive the continuous time limit of the relevant t-statistic based on different estimates of the standard error of the estimate. Our analysis permits us to address size and power issues with respect to κ and the sampling interval used. Our theoretical and simulation results show that power is decreasing as κ increases, contrary to the theoretical result of [Journal of Empirical Finance 8 (2001) 459] based on the [Annals of Mathematical Statistics 31 (1960) 276] approximate slope analysis. Also, power depends much more on the span of the data than on the number of observations per se. The issue of size distortions for commonly used procedures is also addressed.

JEL classification: C22; C53; G14
Keywords: Market efficiency; Long horizon regressions; Stock returns; Forecasting; Power functions

* Corresponding author. Tel.: +1-617-353-3026; fax: +1-617-353-4449.
E-mail addresses: perron@bu.edu (P. Perron), vodounoc@caramail.com (C. Vodounou).

doi:10.1016/j.jempfin.2003.04.001

1. Introduction

Since the seminal work of Bachelier (1900) and Fama (1965), the random walk hypothesis has been an integral part of theories pertaining to financial time series. In particular, this hypothesis can be framed in a statistical framework to model the concept of market efficiency in the sense that the best predictor of future prices are current ones (Fama, 1970, 1991). We consider the statistical properties of tests for market efficiency, more specifically return predictibility, in the context of long horizon returns when, under the alternative hypothesis, a substantially correlated series such as the dividend / price ratio helps to forecast returns.

Many studies have considered testing the null hypothesis of a random walk for prices against a variety of alternative hypotheses. Some specify a time series representation different from the random walk (e.g., a stationary autoregressive process or the sum of a permanent and transitory component; see Fama and French, 1988a; Poterba and Summers, 1988; Lo and Mackinlay, 1988). Others attempt to assess whether some regressors have a predictive power for returns at some horizon (e.g., lagged returns, interest rate or the dividend–price ratio; Hansen and Hodrick, 1980; Fama and French, 1988a,b). A common conclusion is that the market efficiency hypothesis is rejected when using long horizons returns, which implies a mean-reverting behavior for prices. In particular, evidence has been put forward to the effect that long horizon returns (3–10 years) are negatively correlated (Fama and French, 1988a; Poterba and Summers, 1988), and that the dividend–price ratio has a positive effect on excess returns (Rozeff, 1984; Shiller, 1984; Campbell and Shiller, 1988). According to Fama and French (1988b), this effect is weak for returns computed over short horizons (1–3 months) but substantial for long horizon returns (2–4 years).

When using statistics based on K-period returns, simulations show that the standard asymptotic normal distribution provides a poor approximation to the exact finite sample distribution, which leads to tests with distorted sizes. Another concern is the power of the tests, especially with respect to the aggregation parameter K.

Richardson and Stock (1989) have proposed, as a solution to the size problem, an asymptotic framework whereby the aggregation parameter K of returns is a fixed fraction of total sample size such that $K/T \rightarrow \kappa$ as $T \rightarrow \infty$. The limit distribution for the autocorrelation coefficient of K-period returns (used by Fama and French, 1988a) and the variance ratio (used by Lo and Mackinlay, 1988) are then nonstandard and functions of Wiener processes. The quality of the finite sample approximations provided by this asymptotic framework is good according to simulation results reported in Richardson and Stock (1989).

Campbell (2001) analyzes standard regressions with K-periods returns as the dependent variable (and some other variations) assuming that the relevant regressor (e.g., the dividend–price ratio) follows an AR(1) process. He investigates the power function of the relevant t-statistic using the slope approximation proposed by Bahadur (1960) and Geweke (1981) and shows that power should increase with K. Richardson and Smith (1991) use the same approach but consider lagged returns as the relevant regressor. Unlike Richardson and Stock (1989), Campbell (2001) does not consider the case where K increases such that $K/T \to \kappa$ as $T \to \infty$, but imposes that K does not exceed $T/2$.

In this paper, we extend the asymptotic framework with $K/T \to \kappa$ analyzed by Richardson and Stock (1989) with a continuous-time model that has a discrete-time representation similar to that used by Campbell (2001). As in the studies mentioned above, we consider regressions with one or K-period returns as the dependent variable and for which the explanatory variable, the dividend–price ratio, is modeled as a nearly integrated AR(1) process. To explore the properties of long horizon returns, we consider the t-statistic for the null hypothesis that the regressor has no explanatory power on K-period returns, following the approach used by Fama and French (1988b). We derive the limit distribution of several variants of this statistic under the null hypothesis of unpredictable returns and under a sequence of local alternatives where the price–dividend ratio has predictive power. Under the null hypothesis, we show via simulations that the limit distributions are good approximations to the finite sample distributions, confirming results in Richardson and Stock (1989). Our analysis of power reveals that it decreases as κ increases. Hence, one conclusion is that we show no advantage in using regressions with long horizon returns over single period returns for the type of alternatives considered. This important feature is contrary to the theoretical result of Campbell (2001) based on the Bahadur's (1960) approximate slope analysis. It is, however, consistent with his simulation results. It is also contrary to those obtained for the variance ratio where power initially increases with κ and then decreases in which case there is an optimal value of κ that will maximize power, this value depending on various parameters of the data generating process (see Perron and Vodounou, 2000).

The rest of this paper is structured as follows. Section 2 describes the relevant model adopted to study the properties of the statistics stating the relevant null and alternative hypotheses and the asymptotic framework adopted. In Section 3, we derive the asymptotic distributions of the t-statistic in regressions with one-period returns of length h as the dependent variable when this sampling interval h converges to zero as the sample size increases. We then consider, in Section 4, the limit of the relevant t-statistic in regressions with K-period returns of length h as the dependent variable using several methods to construct the standard errors of the estimate. Section 5 considers simulation experiments to assess: (i) the adequacy of the asymptotic distribution as approximations to the finite sample distributions; (ii) the asymptotic and finite sample power of the tests. In Section 6, we use our theoretical results to reassess the empirical findings of Fama and French (1988b). Section 7 presents brief concluding remarks. Appendix A provides some technical derivations.

2. Model and testable hypothesis

2.1. Continuous-time model

We consider two continuous-time stochastic processes, $P(t)$ and $X(t)$, where $P(t)$ denotes the logarithm of the price of an asset or a portfolio at time t and $X(t)$ represents a variable such as the dividend–price ratio. We let $Z(t) = (P(t), X(t))'$ and assume that $Z(t)$ is generated by the following diffusion process:

$$dZ(t) = AZ(t)dt + dW(t); \tag{1}$$

with $Z(0) = C$, arbitrary constants,

$$A = \begin{bmatrix} 0 & \beta \\ 0 & \gamma \end{bmatrix},$$

and where $W = (W_1, W_2)'$ is a two-dimensional standard Wiener process with covariance

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & \tau \end{bmatrix}. \tag{2}$$

Note that the process $X(t)$ is stationary provided $\gamma < 0$. The solution to the stochastic differential equation (Eq. (1)) is (e.g., Arnold, 1974):

$$Z(t) = \exp(At)Z(0) + \int_0^t \exp((t-s)A)dW(s), \tag{3}$$

with

$$\exp(uA) = \begin{bmatrix} 1 & \frac{\beta}{\gamma}(\exp(\gamma u) - 1) \\ 0 & \exp(\gamma u) \end{bmatrix}.$$

Assuming that $Z(t)$ is observed over the time interval $[0,N]$, and defining the sampling interval h by $Th = N$, we can obtain the exact discrete-time representation $Z_{th} = (P_{th}, X_{th})'$ of $Z(t)$ which is given by the following autoregressive process of order one:

$$Z_{th} = \exp(hA)Z_{(t-1)h} + u_{th}, \qquad t = 1, \ldots, T,$$

$$u_{th} = \int_{(t-1)h}^{th} \exp((th-r)A)dW(r). \tag{4}$$

The random component u_{th} is i.i.d. $N(0, \Omega_h)$ with

$$\Omega_h = \int_0^h \exp(sA)\Sigma\exp(sA)'\, ds \equiv \begin{bmatrix} \Omega_h^{11} & \Omega_h^{12} \\ \Omega_h^{21} & \Omega_h^{22} \end{bmatrix},$$

where

$$\Omega_h^{11} = h + 2\rho h \frac{\beta}{\gamma}\left(\frac{\exp(\gamma h) - 1}{\gamma h} - 1\right) + \tau h \frac{\beta^2}{\gamma^2}\left(\frac{\exp(2\gamma h) - 4\exp(\gamma h) + 3}{2\gamma h} + 1\right),$$

$$\Omega_h^{12} = \Omega_h^{21} = \rho \frac{\exp(\gamma h) - 1}{\gamma} + \frac{\tau\beta}{2}\left(\frac{\exp(\gamma h) - 1}{\gamma}\right)^2,$$

$$\Omega_h^{22} = \tau \frac{\exp(2\gamma h) - 1}{2\gamma}.$$

We define returns over a period of length h by $R_{th} = P_{th} - P_{(t-1)h} = (1 - L)P_{th}$ with L the lag operator such that $Lx_t = x_{t-1}$. Using the notation $u_{th} = (\epsilon_{th}, v_{th})'$ and $\alpha_h = \beta(\exp(\gamma h) - 1)/\gamma$, we obtain from Eq. (3), the following discrete-time model for returns and the dividend–price ratio:

$$R_{th} = \alpha_h X_{(t-1)h} + \epsilon_{th},$$

$$X_{th} = \exp(\gamma h) X_{(t-1)h} + v_{th}. \tag{5}$$

The system (5) is the reference model that we shall use throughout. For a fixed sampling interval, it implies that the univariate process for returns, R_{th}, is an ARMA(1,1) consistent with the idea that asset prices have permanent and transitory components (e.g., Poterba and Summers, 1988; Campbell, 2001). It also implies that conditional on information available at time t, I_{th}, expected returns are given by $E(R_{(t+1)h} | I_{th}) = \alpha_h X_{th}$. Hence, expectations of future returns are affected by the dividend–price ratio.

The null hypothesis of market efficiency being $\{\beta = 0\}$ in model (1), we can specify it in terms of the parameter α_h in Eq. (5) with H_0: $\{\alpha_h = 0\}$. Hence, we shall be interested in testing H_0 against the alternative hypothesis H_1: $\{\alpha_h \neq 0\}$ in the regression:

$$R_{th} = \alpha_h X_{(t-1)h} + \epsilon_{th}. \tag{6}$$

Letting $c = \gamma N$ and $g^* = \beta N$, we can write $= g^*(\exp(c/T) - 1)/c \simeq g^*/T$. Hence, in the asymptotic framework where $T \to \infty$ with N fixed, we can interpret α_h as a sequence of local alternatives with noncentrality parameter g^*. Without loss of generality, we consider unilateral local alternatives with $g^* > 0$ which accords with the empirical result that the dividend–price ratio has a positive effect on returns (Fama and French, 1988b). To test the null hypothesis, we consider in the following sections, the t-statistic for testing $\alpha_h = 0$ in regression (6) estimated by OLS and its extension to K-periods returns. For simplicity, we suppose that $Z(0) = 0$.

We end by noting that the Gaussian assumption on the errors which follows from taking a discrete time approximation to a continuous-time model is not restrictive. The same asymptotic results will hold allowing a general class of processes for the errors of the discrete time model. What is needed are conditions on the discrete time errors ϵ_{th} and v_{th} such that the results stated in Lemma A.1 in the appendix hold. Such conditions can allow nonnormal processes with some forms of heteroskedasticity.

3. Case with one-period returns of length h

A natural statistic to test the relevant null hypothesis discussed above is the t-statistic defined by $t_{\hat{\alpha}_h} = \hat{\alpha}_h / s(\hat{\alpha}_h)$ where $s(\hat{\alpha}_h)$ is the standard error of the OLS estimate, $\hat{\alpha}_h$, of α_h. The asymptotic framework adopted here is to fix the span of the data N and let T increase to infinity; hence, we let the sampling interval decrease to zero at the same rate as T increases. This framework was found to yield useful approximations in the context of integrated or nearly integrated data. For some applications see, e.g., Bergstrom (1984), Phillips (1987) and Perron (1991a,b).

Proposition 1. *Let $\delta = \rho/\sqrt{\tau}$ be the limit of the correlation coefficient of ϵ_{th} and v_{th} when h converges to 0, keeping N fixed. Under the specifications of model (5), we have under H_0:*

$$t_{\hat{\alpha}_h} \Rightarrow \frac{\int_0^1 J_2(r)\mathrm{d}W_{12}(r)}{\left[\int_0^1 J_2(r)^2 \mathrm{d}r\right]^{1/2}} \equiv \sqrt{1-\delta^2}\,\frac{\int_0^1 J_2(r)\mathrm{d}W_1(r)}{\left[\int_0^1 J_2(r)^2 \mathrm{d}r\right]^{1/2}} + \delta \frac{\int_0^1 J_2(r)\mathrm{d}W_2(r)}{\left[\int_0^1 J_2(r)^2 \mathrm{d}r\right]^{1/2}} \equiv Z(c,\delta); \tag{7}$$

and under a sequence of local alternatives H_1: $\{\alpha_h = g^*(\exp(c/T) - 1)/c\}$:

$$t_{\hat{\alpha}_h} \Rightarrow g\left[\int_0^1 J_2(r)^2 \mathrm{d}r\right]^{1/2} + \frac{\int_0^1 J_2(r)\mathrm{d}W_{12}(r)}{\left[\int_0^1 J_2(r)^2 \mathrm{d}r\right]^{1/2}}, \tag{8}$$

where $J_2(r) = \int_0^r \exp(c(r-s))\mathrm{d}W_2(s)$, $W_{12}(r) = \sqrt{1-\delta^2}W_1(r) + \delta W_2(r)$, W_1 and W_2 are independent Wiener processes and $g = \beta N\sqrt{\tau}$ with $\beta > 0$.

Remark 1. If there was a constant in regression (6), the distributions (7) and (8) in Proposition 1 would be the same with $J_2(r)$ replaced by its demeaned counterpart $J_2^*(r) = J_2(r) - \int_0^1 J_2(r)\mathrm{d}r$. The qualitative results reported below about size and power of the tests are qualitatively similar with or without a fitted intercept. Hence, for simplicity, we shall analyze the no intercept case, except when references to particular applications are concerned.

The distribution (7) is nonstandard and is a linear combination of a standard $N(0,1)$, i.e. the component $\int_0^1 J_2(r)\mathrm{d}W_1(r)/[\int_0^1 J_2(r)^2\mathrm{d}r]^{1/2}$ (e.g., Elliott and Stock, 1994), and of the variable $\int_0^1 J_2(r)\mathrm{d}W_2(r)/[\int_0^1 J_2(r)^2\mathrm{d}r]^{1/2}$ which is an extension to the near-integrated context of the so-called Dickey–Fuller distribution that obtains when $c = 0$.

Tests based on the $N(0,1)$ distribution would be appropriate when $\delta = 0$ for any value of c. However, when δ is nonzero, the normal distribution becomes a bad approximation to the exact distribution of $t_{\hat{\alpha}_h}$. This concurs with the results of Elliott and Stock (1994, Table 1) who show negligible size distortions from using the $N(0,1)$ when $\delta = 0$ but considerable

Table 1
Quantiles of the asymptotic distribution of $t_{\hat{\alpha}_h}$ under H_0

	δ	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
$c=-5$	−0.9	−2.00	−1.66	−1.35	−1.00	0.26	1.46	1.81	2.14	2.49
	−0.7	−2.13	−1.76	−1.43	−1.06	0.18	1.46	1.81	2.10	2.50
	−0.5	−2.17	−1.79	−1.48	−1.12	0.14	1.41	1.77	2.06	2.43
	−0.3	−2.26	−1.89	−1.58	−1.23	0.08	1.37	1.74	2.04	2.37
	0.3	−2.44	−2.03	−1.73	−1.37	−0.08	1.21	1.58	1.88	2.22
	0.5	−2.39	−2.05	−1.75	−1.37	−0.11	1.16	1.56	1.82	2.21
	0.9	−2.48	−2.15	−1.84	−1.46	−0.23	1.02	1.35	1.65	2.01
$c=0$	−0.9	−2.03	−1.67	−1.32	−0.90	0.43	1.59	1.93	2.20	2.54
	−0.7	−2.09	−1.70	−1.36	−0.99	0.29	1.54	1.89	2.18	2.49
	−0.5	−2.14	−1.74	−1.45	−1.08	0.23	1.47	1.83	2.16	2.55
	−0.3	−2.18	−1.86	−1.56	−1.18	0.12	1.39	1.74	2.07	2.39
	0.3	−2.45	−2.09	−1.80	−1.43	−0.14	1.12	1.49	1.83	2.19
	0.5	−2.50	−2.17	−1.86	−1.49	−0.21	1.08	1.45	1.75	2.10
	0.9	−2.58	−2.21	−1.92	−1.59	−0.41	0.92	1.29	1.61	2.01
	$N(0,1)$	−2.33	−1.96	−1.65	−1.28	0.00	1.28	1.65	1.96	2.33

ones when δ is nonzero; for example, when $T=50$, $\delta=-0.9$ and the autoregressive coefficient is $\alpha_h=0.95$, the exact size of the test of H_0 against a unilateral alternative is 1% in the left tail and 22% in the right tail (for a nominal size of 5%). Similar distortions occur for different values of α_h, though the distortions become smaller as α_h approaches 0.

The distribution (7) suggests that when $\delta=1$ and $c=0$, using the Dickey–Fuller critical values would be adequate. Elliott and Stock (1994) consider c as a nuisance parameter and propose a sequential procedure which first does a pre-test to decide the order of integration of the regressor and then test the null hypothesis. According to their results, this procedure does not eliminate size distortions since the critical value in the pre-test are fixed and, hence, the Dickey–Fuller test does not select the correct order of integration consistently.

Our aim here is to study the adequacy of the asymptotic distribution as an approximation to the finite sample distribution. To that effect we treat the parameters δ and c as known. We present in Table 1 the quantiles of the distribution of the variable specified in Eq. (7) and in Table 2 the exact size of a test of H_0 against a unilateral alternative $\{\alpha_h>0\}$ obtained using the asymptotic critical values. To simulate the limit distribution, the integrals $\int_0^1 J_2(r)\mathrm{d}W_{12}(r)$ and $\int_0^1 J_2(r)^2\mathrm{d}r$ are approximated, respectively, by $T^{-1}\sum_{t=1}^{T} X_{t-1}z_t$ and $T^{-2}\sum_{t=1}^{T} X_{t-1}^2$ with $T=800$, $X_t=\exp(c/T)X_{t-1}+v_t$ and $z_t=\sqrt{1-\delta^2}u_t+\delta v_t$, u_t and v_t being independent $N(0,1)$ random variables. All simulations are based on 10,000 replications.

Table 2
Exact size of the test $t_{\hat{\alpha}_h}$ using asymptotic critical values for a nominal size 5%

T	$\delta=0.0$		$\delta=-0.5$		$\delta=-0.9$	
	$c=-5$	$c=0$	$c=-5$	$c=0$	$c=-5$	$c=0$
10	0.09	0.09	0.09	0.09	0.08	0.09
25	0.07	0.07	0.07	0.07	0.07	0.07
50	0.06	0.06	0.06	0.06	0.06	0.06
100	0.05	0.05	0.05	0.05	0.06	0.05

As expected from Proposition 1, the results in Table 1 show that the limit distribution (7) is nonnormal unless $\delta = 0$. When δ is negative, it has a shorter left tail than the normal (more so as δ decreases) and vice versa when δ is positive. The shifts induced by variations in δ appear symmetric for corresponding positive and negative values of δ.

Concerning the adequacy of the asymptotic distribution (7) as an approximation to the finite sample distribution, the results show that it is satisfactory especially when T reaches 50 or 100 (see Table 2). When $T = 10$ or 25, the approximation is less satisfactory and the size distortions become more noticeable when δ and c increase.

The asymptotic framework used also permits analyzing the limit distribution of the test when the span of the data N increases. Since $c = \gamma N$ and $T = N/h$, the limit distribution as $c \to \infty$ for $\gamma > 0$ corresponds to that when h is fixed or to that where $h \to 0$ and $N \to \infty$. From Phillips (1987, Lemma 2) and Perron (1991a, Lemma A.2), we can show that if c diverges to $\pm\infty$ the limit distribution of $t_{\hat{\alpha}_h}$ is that of a $N(0,1)$.

The limit distribution (8) also permits us to evaluate the asymptotic power function of the test under a sequence of local alternatives of the form $\alpha_h = g(\exp(c/T) - 1)/c$ with $\beta > 0$. For a size μ, this power function is defined by:

$$\lim_{T\to\infty} P_{T,N}(\alpha_h) = \lim_{h\to 0} \Pr\{t_{\hat{\alpha}_h} > \lambda_{1-\mu}\} = \Pr\left\{ g\left[\int_0^1 J_2(r)^2 \mathrm{d}r\right]^{1/2} + Z(c,\delta) > \lambda_{1-\mu} \right\},$$

where $\lambda_{1-\mu}$ denotes the quantile of order $1-\mu$ of the distribution of $Z(c,\delta)$ and $g = \beta N$. Via the coefficient g, the power function depends on the span of the data N and on the parameters β and τ. It also depends in obvious ways on c and δ. For a fixed δ, we can see that the power converges to the size of the test when $N \to 0$; if $N \to \infty$, the test is consistent. On the other hand, we can also evaluate the power function for different values of β, τ, c and δ keeping N fixed.

To assess the relevance of these theoretical results, we analyze the exact power function $P_{T,N}(\alpha_h)$ of a unilateral test of size 5% of the null hypothesis $\alpha_h = 0$ versus an alternative hypothesis that $\alpha_h > 0$. To that effect, we consider the values $N = 8$, 16, 32, 64, 128, 256, 512 and $T = 8$, 16, 32, 64, 128, 256, 512. For the nuisance parameters τ and $\delta = \rho/\sqrt{\tau}$ in model (5), we specify $\gamma = -0.02$, $\tau = 1$ and $\delta = 0$, -0.5, -0.9. The qualitative results are the same with alternative specifications.

Critical values of the statistic $t_{\hat{\alpha}_h}$ are obtained from 10,000 replications for each pair (N,T). Under the alternative hypothesis defined implicitly by $\beta = 0.1$, we have $\alpha_h = \beta(\exp(\gamma N/T) - 1)/\gamma$, and the power is evaluated using 2000 replications. The results are presented in Table 3. They show that for any fixed δ, the power increases with T when N is fixed. If N is small the power function is close to 5%, the size of the test, as suggested by the theoretical results. The power increases most notably when N increases for a given fixed T. For example, when $T = 16$ and $\delta = 0$, the exact power is 0.70 when $N = 64$ and 0.89 when $N = 128$. On the other hand, when $N = 16$, it is 0.23 when $T = 64$ and 0.26 when $T = 128$. These results show that power depends much more on the span of the data than on the number of observations per se (see also Shiller and Perron, 1985; Perron, 1991c). Looking at the power as a function of δ, different outcomes occur

Table 3
Power of the statistic $t_{\hat{\alpha}_h}$ ($\alpha_h = \beta(\exp(\gamma h) - 1)/\gamma$, $\beta = 0.1$, $\gamma = -0.02$)

$N \backslash T$	8	16	32	64	128	256	512
$\delta = 0.0$							
8	0.11	0.11	0.12	0.13	0.13	0.12	0.14
16	0.18	0.22	0.23	0.23	0.26	0.25	0.26
32	0.37	0.44	0.46	0.47	0.51	0.44	0.47
64	0.54	0.70	0.74	0.77	0.77	0.79	0.79
128	0.71	0.89	0.93	0.95	0.96	0.97	0.97
256	0.72	0.95	0.99	1.00	1.00	1.00	1.00
512	0.59	0.95	1.00	1.00	1.00	1.00	1.00
$\delta = -0.5$							
8	0.11	0.09	0.10	0.11	0.11	0.11	0.12
16	0.18	0.21	0.20	0.18	0.23	0.20	0.23
32	0.42	0.48	0.50	0.50	0.53	0.45	0.49
64	0.69	0.81	0.83	0.86	0.86	0.88	0.86
128	0.85	0.96	0.99	0.99	0.99	0.99	1.00
256	0.87	0.99	1.00	1.00	1.00	1.00	1.00
512	0.72	0.99	1.00	1.00	1.00	1.00	1.00
$\delta = -0.9$							
8	0.10	0.08	0.08	0.08	0.10	0.09	0.08
16	0.17	0.17	0.18	0.16	0.18	0.14	0.19
32	0.56	0.60	0.61	0.61	0.58	0.57	0.57
64	0.92	0.97	0.97	0.97	0.98	0.98	0.98
128	0.99	1.00	1.00	1.00	1.00	1.00	1.00
256	0.98	1.00	1.00	1.00	1.00	1.00	1.00
512	0.86	1.00	1.00	1.00	1.00	1.00	1.00

according to whether N is small or large. Indeed, when N is small ($N \leq 16$) the power function decreases as δ increases (in absolute value). On the other hand, if N is large ($N \geq 32$) the reverse holds.

4. Extensions to *K*-periods returns of length *h*

An empirical result that has received considerable attention relates to the fact that the dividend/price ratio has a considerable predictive power on returns over long-term horizons (e.g., Fama and French, 1988b; Campbell, 2001). The type of regression on which this result is based specifies K-periods returns as the dependent variable (defined as the log differences of prices over K-periods) and the lagged dividend/price ratio as the regressor. The important parameter K is usually specified in terms of annual units; for example, Fama and French (1988a) consider K ranging from 1 to 10 years and Fama and French (1988b) from 2 to 4 years with 720 monthly observations.

Our intent in this section is to study the properties of tests in the type of regression described above. To that effect, assuming without loss of generality a zero intercept, we consider the following regression estimated by OLS with $t=1,\ldots,T-K+1$:

$$\sum_{i=0}^{K-1} R_{(t+i)h} = \hat{\alpha}_h(K)X_{(t-1)h} + \hat{\xi}_{th}(K). \tag{9}$$

The null hypothesis of market efficiency is specified as H_0: $\{\alpha_h(K)=0\}$ and the alternative hypothesis as H_1: $\{\alpha_h(K)>0\}$. From model (5), the population values of $\alpha_h(K)$ and $\xi_{th}(K)$ are

$$\alpha_h(K) = \alpha_h(\exp(\gamma Kh) - 1)/(\exp(\gamma h) - 1),$$

and

$$\xi_h(K) = \alpha_h \sum_{i=1}^{K-1}\sum_{j=0}^{i-1} \exp(\gamma h(i-j-1))v_{(t+j)h} + \sum_{i=0}^{K-1} \epsilon_{(t+i)h}.$$

In regression (9), the returns are constructed using overlapping observations which induces correlation in the errors. Indeed, under the null hypothesis, we have $\xi_{th}(K) = \sum_{i=0}^{K-1} \epsilon_{(t+i)h}$ which have nonzero correlation up to lag $(K-1)$.

Here, we focus on the case where both K and T increase to infinity with κ $(0 \le \kappa < 1)$ as the limit value of the ratio K/T. The special case with $\kappa=0$ can be considered as the base case where K is fixed and relatively small compared to T.

4.1. Statistic based on a HAC variance estimate

A consistent estimate of the asymptotic variance of $(T-K+1)^{1/2}\hat{\alpha}_h(K)$ is given by:

$$(T-K+1)\hat{V}(\hat{\alpha}_h(K)) = \frac{\sum_{j=-K+1}^{K-1} w_T(j/K)\hat{R}(j)}{\left\{(T-K+1)^{-1}\sum_{t=1}^{T-K+1} X_{(t-1)h}^2\right\}^2}, \tag{10}$$

where

$$\hat{R}(j) = (T-K+1)^{-1}\sum_{t=j+1}^{T-K+1}(X_{(t-1)h}\hat{\xi}_{th}(K))(X_{(t-1-j)h}\hat{\xi}_{(t-j)h}(K)),$$

and $\hat{\xi}_{th}(K) = \sum_{i=0}^{K-1} R_{(t+i)h} - \hat{\alpha}_h(K)X_{(t-1)h}$, with $\hat{\alpha}_h(K)$ the OLS estimate of $\alpha_h(K)$ in regression (9). The term $w_T(j/K)$ is a weight function which for illustration purposes we choose to be the Bartlett triangular kernel defined by $w_T(x)=1-|x|$ for $|x| \le 1$, as proposed in econometrics by Newey and West (1987). In Eq. (10), the limit of the component $\sum_{j=-K+1}^{K-1} w_T(j/T)\hat{R}(j)$ corresponds to (2π) times the spectral density function at frequency zero of the stochastic process $X_{(t-1)h}\xi_{th}(K)$. The statistic considered is then the modified t-ratio defined by

$$t^{\mathrm{NW}}_{\alpha_h(K)} = \{\hat{\alpha}_h(K)/(\hat{V}\hat{\alpha}_h(K))\}^{1/2}.$$

To analyze its asymptotic properties, we start with the following lemma.

Lemma 1. *Let* (R_{th},X_{th}) *be generated by Eq. (5),* $(R^*_{th},X^*_{th})=(R_{th},X_{th})/h^{1/2}$, $R^*_{th}(K)=\sum_{i=0}^{K-1} R^*_{(t+i)h}$ *and* $\epsilon^*_{th}(K)=\sum_{i=0}^{K-1}\epsilon^*_{(t+i)h}$. *We suppose that* $K\to\infty$ *and* $K/T\to\kappa$ *with h converging to 0 (and* $T\to\infty$*). Then,*

$$(a)\ T^{-1}K^{-1}\sum_{t=1}^{T-K+1} X^*_{(t-1)h}\epsilon^*_{th}(K)\Rightarrow\sqrt{\tau}\int_0^1\left\{\int_{\kappa p}^{1-\kappa+\kappa p} J_2(r-\kappa p)\mathrm{d}W_{12}(r)\right\}\mathrm{d}p;$$

$$(b)\ T^{-2}\sum_{t=1}^{T-K+1} X^{*2}_{(t-1)h}\Rightarrow\tau\int_0^{1-\kappa} J_2(r)^2\mathrm{d}r;$$

$$(c)\ T^{-1/2}R^*_{[Tr]h}(K)\Rightarrow\beta N\sqrt{\tau}\int_r^{r+\kappa} J_2(s)\mathrm{d}s+W_{12}(r+\kappa)-W_{12}(r)\ \text{ for } 0\le r\le 1;$$

$$(d)\ T^{-2}K^{-1}\sum_{t=1}^{T-K+1}\sum_{i-0}^{K-1} X^*_{(t-1)h}X^*_{(t-1+i)h}\Rightarrow\frac{\tau}{\kappa}\int_0^{1-\kappa}\int_r^{r+\kappa} J_2(r)J_2(s)\mathrm{d}s\mathrm{d}r;$$

$$(e)\ \hat{V}(\hat{\alpha}_h(K))\Rightarrow\frac{1}{\tau}\frac{\int_{-\kappa}^{\kappa}\left(1-\frac{|u|}{\kappa}\right)\left[\int_u^{1-\kappa} J_2(r)J_2(r-u)H_{12}(r)H_{12}(r-u)\mathrm{d}r\right]\mathrm{d}u}{\left\{\int_0^{1-\kappa} J_2(r)^2\mathrm{d}r\right\}^2};$$

where $J_2(r)\int_0^r \exp(c(r-s))\mathrm{d}W_2(s)$, $H_{12}(r)=\beta N\sqrt{\tau}\int_r^{r+\kappa} J_2(s)\mathrm{d}s+H^0_{12}(r)$, $W_{12}(r)=\sqrt{1-\delta^2}\,W_1(r)+\delta W_2(r)$, *$W_1(r)$ and $W_2(r)$ are two independent Wiener processes, and*

$$H^0_{12}(r)=W_{12}(r+\kappa)-W_{12}(r)-J_2(r)\frac{\int_0^{\kappa}\left[\int_s^{1-\kappa+s} J_2(r-s)\mathrm{d}W_{12}(r)\right]\mathrm{d}s}{\int_0^{1-\kappa} J_2(r)^2\mathrm{d}r}.$$

Replacing $H_{12}(x)$ by $H^0_{12}(x)$ in part (e) of Lemma 1 shows that, under the null hypothesis, the estimate of the asymptotic variance of $\hat{\alpha}_h(K)$, $\hat{V}(\hat{\alpha}_h(K))$, does not converge to a constant but rather to a random variable whenever κ is nonzero. The limit distribution of $\alpha_h(K)$, $t^{\mathrm{NW}}_{\alpha_h(K)}$ and of the R^2 of the regression (9) are given in the following proposition, which follows directly from Lemma 1.

Proposition 2. *Under the hypotheses of Lemma 1, we have under H_0:*

$$\text{(a) } \hat{\alpha}_h(K) \Rightarrow \frac{1}{\sqrt{\tau}} \frac{\int_0^{\kappa} \int_s^{1-\kappa+s} J_2(r-s)\mathrm{d}W_{12}(r)\mathrm{d}s}{\int_0^{1-\kappa} J_2(r)^2 \mathrm{d}r} \tag{11}$$

$$\text{(b) } t^{\mathrm{NW}}_{\alpha_h(K)} \Rightarrow \frac{\int_0^{\kappa} \int_s^{1-\kappa+s} J_2(r-s)\mathrm{d}W_{12}(r)\mathrm{d}s}{[Z_1^{\mathrm{NW}}(c,\kappa)]^{1/2}} \tag{12}$$

$$\text{(c) } R^2 \Rightarrow \frac{\left\{\int_0^{\kappa} \int_s^{1-\kappa+s} J_2(r-s)\mathrm{d}W_{12}(r)\mathrm{d}s\right\}^2}{\int_0^{1-\kappa} J_2(r)^2 \mathrm{d}r \int_0^{1-\kappa} [W_{12}(r+\kappa) - W_{12}(r)]^2 \mathrm{d}r} \tag{13}$$

with

$$Z_1^{\mathrm{NW}}(c,\kappa) = \int_{-\kappa}^{\kappa} \left(1 - \frac{|u|}{\kappa}\right) \left[\int_u^{1-\kappa} J_2(r)J_2(r-u)H_{12}^0(r)H_{12}^0(r-u)\mathrm{d}r\right] \mathrm{d}u \tag{14}$$

and where $W_1(r)$, $W_2(r)$, $W_{12}(r)$ and $H_{12}^0(r)$ are as defined in Lemma 1.

Proposition 2 shows that the asymptotic distribution of $\hat{\alpha}_h(K)$ is a nondegenerate random variable for all $\kappa \neq 0$. Hence, $\hat{\alpha}_h(K)$ is not a consistent estimate of $\hat{\alpha}_h(K)$. However, if $\kappa = 0$, $\hat{\alpha}_h(K)$ then converges to 0. Proposition also shows that the limit distribution of $t^{\mathrm{NW}}_{\alpha_h(K)}$ is nonstandard. In particular, when $\kappa = 0$, the limit distribution (12) is identical to that given by Eq. (7). Hence, the limiting results given in proposition can be viewed as a generalization of the case where K is fixed or relatively small compared to T.

Campbell (2001) analyzed the goodness of fit measure R^2 in the same context and argues that this statistics can converge to 0 if $K \to \infty$. From Eq. (13), this result is obtained when $\kappa = 0$. We can also show that in this case TR^2 has the same asymptotic distribution as $t^2_{\hat{\alpha}_h}$ under H_0; and that this distribution is a chi-square with one degree of freedom when $\delta = 0$. When $\kappa \neq 0$, the R^2 has a nondegenerate limit distribution, a result which contrasts with those of Campbell (2001).

To analyze the asymptotic power of the test, we suppose that the processes (R_{th}, X_{th}) are generated by Eq. (5) with $\alpha_h = \beta(\exp(\gamma h) - 1)/\gamma$ and $\beta > 0$. From Lemma 1, the asymptotic distribution of $t^{\mathrm{NW}}_{\alpha_h(K)}$ under this alternative is given by:

$$t^{\mathrm{NW}}_{\alpha_h(K)} \Rightarrow \frac{A(c,\kappa,\beta,\rho)}{[G(c,\kappa,\beta,\rho)]^{1/2}}, \tag{15}$$

with

$$A(c,\kappa,\beta,\rho) = g\int_0^{1-\kappa}\int_r^{r+\kappa} J_2(r)J_2(s)\mathrm{d}s\mathrm{d}r + \int_0^{\kappa}\int_s^{1-\kappa+s} J_2(r-s)\mathrm{d}W_{12}(r)\mathrm{d}s$$

$$G(c,\kappa,\beta,\rho) = \int_{-\kappa}^{\kappa}\left(1-\frac{|u|}{\kappa}\right)\left[\int_u^{1-\kappa} J_2(r)J_2(r-u)H_{12}(r)H_{12}(r-u)\mathrm{d}r\right]\mathrm{d}u$$

and $g = \beta N\sqrt{\tau}$. For a nominal size μ and any nonzero κ, the asymptotic power function is:

$$\lim_{T,K\to\infty} P_{T,K,N}(\alpha_h(K)) = \lim_{h\to 0}\Pr\left\{t^{\mathrm{NW}}_{\hat{\alpha}_h(K)} > \lambda_{1-\mu}(\kappa)\right\}$$

$$= \Pr\{A(c,\kappa,\beta,\rho)/[G(c,\kappa,\beta,\rho)]^{1/2} > \lambda_{1-\mu}(\kappa)\}$$

where $\lambda_{1-\mu}(\kappa)$ is the $(1-\mu)$% point of the distribution (12). As for the case with one-period returns of length h, the power is equal to the size of the test as N approaches 0, and the test is consistent as N increases. If N is fixed the power will depend, in particular, on κ.

4.2. Statistic using Hansen and Hodrick's (1980) estimate of the variance

Hansen and Hodrick (1980) considered a different estimate of the asymptotic variance of $\hat{\alpha}_h(K)$. With Bartlett weights, it is defined by:

$$(T-K+1)\hat{V}_H(\hat{\alpha}_h(K)) = \frac{\sum_{j=-K+1}^{K-1}\left(1-\frac{|j|}{K}\right)\hat{R}_X(j)\hat{R}_\xi(j)}{\left\{(T-K+1)^{-1}\sum_{t=1}^{T-K+1} X^2_{(t-1)h}\right\}^2} \qquad (16)$$

with

$$\hat{R}_X(j) = (T-K+1)^{-1}\sum_{t=j+1}^{T-K+1} X_{(t-1)h}X_{(t-1-j)h}$$

and

$$\hat{R}_\xi(j) = (T-K+1)^{-1}\sum_{t=j+1}^{T-K+1} \hat{\xi}_{th}(K)\hat{\xi}_{(t-j)h}(K)$$

Under H_0 the estimate $\hat{V}_H(\hat{\alpha}_h(K))$ does not converge to a constant but rather to a random variable whose distribution is described in the appendix. We then deduce that the t-ratio, denoted $t^H_{\hat{\alpha}_h(K)}$, has the following limit distribution under the null hypothesis:

$$t^H_{\alpha_h(K)} \Rightarrow \sqrt{1-\kappa}\,\frac{\int_0^{\kappa}\int_s^{1-\kappa+s} J_2(r-s)\mathrm{d}W_{12}(r)\mathrm{d}s}{[Z_1^H(c,\kappa)]^{1/2}}, \qquad (17)$$

where

$$Z_1^H(c,\kappa)=\int_{-\kappa}^{\kappa}\left(1-\frac{|u|}{\kappa}\right)\left[\int_u^{1-\kappa}J_2(r)J_2(r-u)\mathrm{d}r\right]\left[\int_u^{1-\kappa}H_{12}^0(r)H_{12}^0(r-u)\mathrm{d}r\right]\mathrm{d}u.$$

4.3. Statistic with a standard estimate of the variance

Without taking into account the fact that the errors are correlated under H_0, the estimate of the variance of $\hat{\alpha}_h(K)$ is $\hat{\sigma}^2/\sum_{t=1}^{T-K+1}X_{(t-1)h}^2$ where

$$\hat{\sigma}^2=(T-K+1)^{-1}\sum_{t=1}^{T-K+1}(R_{th}(K)-\hat{\alpha}_h(K)X_{(t-1)h})^2$$

In this case the t-ratio is defined by

$$t_{\alpha_h(K)}^{\mathrm{STD}}=\hat{\alpha}_h(K)\Bigg/\left\{\hat{\sigma}^2\Bigg/\sum_{t=1}^{T-K+1}X_{(t-1)h}^2\right\}^{1/2}.$$

Under the null hypothesis, the statistic $t_{\alpha_h(K)}^{\mathrm{STD}}$ diverges as K increases and $K^{-1/2}t_{\alpha_h(K)}^{\mathrm{STD}}$ has the following limit distribution:

$$K^{-1/2}t_{\alpha_h(K)}^{\mathrm{STD}}\Rightarrow\sqrt{\frac{1-\kappa}{\kappa}}\frac{\int_0^{\kappa}\int_s^{1-\kappa+s}J_2(r-s)\mathrm{d}W_{12}(r)\mathrm{d}s}{[Z_2(c,\kappa)]^{1/2}},\tag{18}$$

with

$$\begin{aligned}Z_2(c,\kappa)=&\int_0^{1-\kappa}[W_{12}(r+\kappa)-W_{12}(r)]^2\mathrm{d}r\int_0^{1-\kappa}J_2(r)^2\mathrm{d}r\\&-\left(\int_0^{\kappa}\int_s^{1-\kappa+s}J_2(r-s)\mathrm{d}W_{12}(r)\mathrm{d}s\right)^2.\end{aligned}$$

This divergence suggests that using $t_{\alpha_h(K)}^{\mathrm{STD}}$ to test market efficiency would result in rejecting the null hypothesis too often.

5. Simulation experiments

Our aim in this section is first to study the asymptotic distributions (12) and (17) as a function of various parameters, in particular κ. Secondly, we analyze the properties of the test statistics $t_{\alpha_h(K)}^{\mathrm{NW}}$ and $t_{\alpha_h(K)}^{\mathrm{STD}}$. The limit distribution of the statistic $t_{\alpha_h(K)}^{\mathrm{H}}$ presents features similar to that of $t_{\alpha_h(K)}^{\mathrm{NW}}$, hence separate results are not presented. We consider how well the asymptotic distributions approximate the finite sample distributions and we investigate the asymptotic and finite sample power (limiting ourselves to the statistic $t_{\alpha_h(K)}^{\mathrm{NW}}$ for the latter).

Table 4
Quantiles of the asymptotic distribution of $t^{\mathrm{NW}}_{\alpha_h(K)}$ under H_0 (multi-periods returns)

	κ	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
$\delta=0.0$										
$c=-5$	0.002	−2.75	−2.32	−1.93	−1.52	−0.00	1.49	1.91	2.27	2.71
	0.050	−3.63	−3.01	−2.39	−1.79	0.02	1.87	2.44	2.96	3.62
	0.125	−5.15	−4.00	−3.13	−2.30	−0.04	2.15	3.03	3.91	5.01
	0.250	−8.40	−6.07	−4.51	−3.12	0.01	3.13	4.31	5.64	7.76
	0.500	−20.48	−14.78	−10.76	−6.88	0.12	7.10	10.47	14.94	20.51
$c=0$	0.002	−2.79	−2.32	−1.94	−1.50	−0.02	1.49	1.91	2.28	2.71
	0.050	−3.68	−2.95	−2.39	−1.79	0.04	1.86	2.42	2.96	3.65
	0.125	−5.24	−3.92	−3.15	−2.30	−0.01	2.22	3.13	3.95	5.06
	0.250	−9.23	−6.71	−4.89	−3.41	−0.04	3.42	4.87	6.51	8.93
	0.500	−25.35	−17.47	−13.02	−8.55	0.14	8.66	12.64	18.04	25.84
$\delta=-0.5$										
$c=-5$	0.002	−2.51	−2.12	−1.76	−1.33	0.15	1.64	2.06	2.44	2.86
	0.050	−3.40	−2.70	−2.16	−1.58	0.20	2.06	2.63	3.29	3.94
	0.125	−4.57	−3.50	−2.69	−1.94	0.17	2.44	3.38	4.23	5.58
	0.250	−7.24	−5.17	−3.84	−2.61	0.27	3.52	5.07	6.65	9.09
	0.500	−19.34	−13.99	−9.82	−6.28	0.55	7.92	12.08	16.58	22.04
$c=0$	0.002	−2.47	−2.05	−1.68	−1.25	0.24	1.76	2.19	2.53	2.99
	0.050	−3.39	−2.65	−2.10	−1.53	0.30	2.14	2.78	3.37	4.09
	0.125	−4.61	−3.46	−2.71	−1.92	0.26	2.65	3.58	4.48	5.94
	0.250	−7.98	−5.60	−4.15	−2.79	0.31	3.91	5.67	7.69	10.47
	0.500	−21.77	−15.31	−11.39	−7.43	0.64	10.57	15.61	21.28	29.40
$\delta=-0.9$										
$c=-5$	0.002	−2.41	−1.95	−1.61	−1.18	0.28	1.72	2.16	2.53	2.94
	0.050	−2.96	−2.37	−1.93	−1.42	0.33	2.23	2.82	3.42	4.12
	0.125	−3.89	−2.98	−2.28	−1.61	0.36	2.71	3.60	4.56	5.75
	0.250	−6.45	−4.40	−3.23	−2.15	0.53	3.91	5.31	7.07	9.44
	0.500	−17.32	−12.44	−9.04	−5.82	0.94	8.35	12.00	16.60	23.59
$c=0$	0.002	−2.35	−1.91	−1.51	−1.05	0.49	1.91	2.29	2.68	3.08
	0.050	−3.07	−2.45	−1.95	−1.32	0.54	2.39	2.95	3.51	4.27
	0.125	−4.31	−3.22	−2.45	−1.65	0.57	2.98	3.92	4.90	6.17
	0.250	−7.02	−4.98	−3.69	−2.44	0.71	4.58	6.36	8.22	11.58
	0.500	−18.75	−13.52	−9.76	−6.60	1.06	12.20	18.28	25.70	35.93

5.1. Properties of the asymptotic distributions

The parameters affecting the limiting distributions are c, κ, δ and τ. For c and δ, we have selected values that are usual in the literature. For c, we consider, following Perron (1991a) and others, $c=-5$, 0 to take into account locally stationary ($c<0$) and integrated ($c=0$) processes.[1] For κ, we consider values covering a wide range, from $\kappa=0.002$ to $\kappa=0.5$ (the case with $\kappa=0$ having been studied in Section 2). The parameter δ being the limit of the correlation coefficient of ϵ_{th} and v_{th} when the sampling interval h converges to

[1] In a previous draft, we also considered the locally explosive case with $c=2$. Since this is a case of less relevance in practice, results are not included.

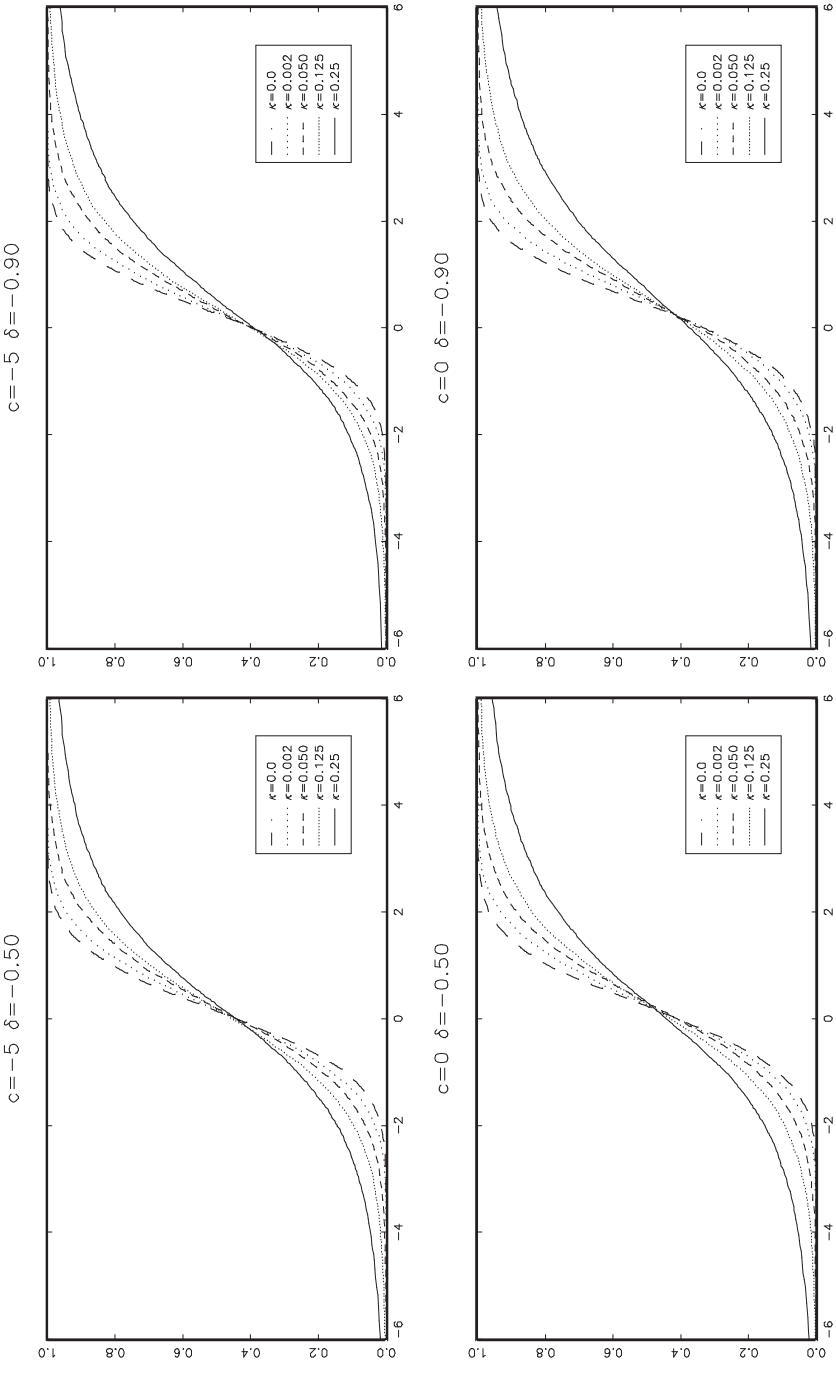

Fig. 1. Asymptotic distribution of $t^{\mathrm{NW}}_{\alpha_\mu(K)}$ under H_0.

0, we have chosen negative values to ensure negatively correlated returns under the alternative hypothesis; the values chosen are $\delta = 0, -0.5, -0.9$ (see, e.g., Elliott and Stock, 1994). Finally, the parameter τ, being interpreted as the limit of the relative variance of $v_{th}/\sqrt{h}$ when h converges to 0, is fixed at 1 without loss of generality. The exact distributions are obtained for different values of T using 10,000 simulations of the process (5). The limiting distributions are also obtained using simulations; $J_2(r)$ is approximated by $T^{-1/2}X_{[Tr]-1}$, $W_{12}(r)$ by $T^{-1/2}\sum_{i=1}^{[Tr]}(\sqrt{1-\delta^2}u_i + \delta v_i)$, the various integrals are constructed by the appropriate normalized sums with $T=1000$, and $X_t = \exp(c/T)X_{t-1} + v_t$ where u_t and v_t are drawn independently from a $N(0,1)$.

Table 4 and Fig. 1 present the asymptotic distribution of the statistic $t^{\mathrm{NW}}_{\alpha_h(K)}$ under the null hypothesis. When $\delta = 0$, the limit distribution is symmetric and centered at 0 for any given values of c and κ but with fatter tails than the normal distribution, the importance of the tails increasing as κ increases. A similar effect is present with respect to the noncentrality parameter c. When $\delta \neq 0$ (Fig. 1), the limit distribution is still characterized by fat tails and we observe a shift to the right as δ increases in absolute value.

Table 5
Quantiles of the asymptotic distribution of the statistic $K^{-1/2}t^{\mathrm{STD}}_{\alpha_h(K)}$

	κ	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%
$\delta = 0.0$										
$c = -5$	0.010	−2.37	−1.95	−1.65	−1.27	0.01	1.26	1.65	1.96	2.29
	0.125	−2.25	−1.89	−1.59	−1.22	−0.02	1.16	1.53	1.87	2.32
	0.250	−2.21	−1.81	−1.49	−1.13	−0.01	1.11	1.45	1.76	2.18
	0.500	−1.80	−1.48	−1.22	−0.90	−0.01	0.91	1.21	1.51	1.87
$c = 0$	0.010	−2.38	−1.99	−1.65	−1.28	−0.00	1.26	1.65	1.97	2.36
	0.125	−2.61	−2.07	−1.72	−1.32	−0.01	1.28	1.71	2.07	2.52
	0.250	−2.72	−2.21	−1.79	−1.34	0.01	1.33	1.80	2.24	2.75
	0.500	−2.48	−1.95	−1.58	−1.19	−0.01	1.18	1.60	1.97	2.58
$\delta = -0.5$										
$c = -5$	0.010	−2.24	−1.84	−1.51	−1.16	0.13	1.39	1.74	2.09	2.50
	0.125	−2.13	−1.70	−1.42	−1.06	0.10	1.27	1.62	1.94	2.36
	0.250	−2.06	−1.64	−1.33	−0.98	0.10	1.20	1.55	1.88	2.24
	0.500	−1.62	−1.30	−1.05	−0.78	0.09	1.04	1.36	1.68	2.08
$c = 0$	0.010	−2.20	−1.81	−1.45	−1.08	0.20	1.46	1.81	2.13	2.50
	0.125	−2.31	−1.90	−1.51	−1.11	0.18	1.49	1.89	2.27	2.74
	0.250	−2.39	−1.93	−1.54	−1.14	0.15	1.53	2.00	2.44	3.06
	0.500	−2.13	−1.68	−1.33	−1.00	0.10	1.42	1.91	2.36	3.01
$\delta = -0.9$										
$c = -5$	0.010	−2.04	−1.70	−1.37	−1.01	0.22	1.47	1.84	2.19	2.53
	0.125	−1.94	−1.52	−1.23	−0.89	0.20	1.32	1.63	1.91	2.32
	0.250	−1.79	−1.44	−1.14	−0.81	0.20	1.25	1.55	1.82	2.21
	0.500	−1.44	−1.17	−0.94	−0.68	0.18	1.12	1.45	1.76	2.11
$c = 0$	0.010	−2.05	−1.67	−1.34	−0.95	0.41	1.60	1.92	2.22	2.55
	0.125	−2.14	−1.71	−1.34	−0.95	0.39	1.61	1.97	2.32	2.70
	0.250	−2.14	−1.72	−1.39	−0.99	0.33	1.65	2.04	2.42	2.94
	0.500	−1.86	−1.49	−1.20	−0.88	0.22	1.69	2.31	2.85	3.53

Table 6

	T	$\delta=0.0$		$\delta=-0.5$		$\delta=-0.9$	
		$c=-5$	$c=0$	$c=-5$	$c=0$	$c=-5$	$c=0$
(a) Exact size of the statistic $t^{NW}_{\hat{\alpha}_h(K)}$ using critical values of the asymptotic distribution for a nominal size 5%							
$\kappa=0.05$	20	0.01	0.01	0.01	0.01	0.01	0.01
	40	0.04	0.04	0.04	0.04	0.03	0.04
	60	0.04	0.04	0.05	0.04	0.04	0.04
	100	0.05	0.05	0.05	0.05	0.05	0.05
	200	0.05	0.05	0.05	0.04	0.05	0.05
$\kappa=0.25$	20	0.05	0.05	0.04	0.05	0.04	0.04
	40	0.05	0.05	0.05	0.05	0.05	0.05
	60	0.05	0.05	0.05	0.05	0.05	0.05
	100	0.05	0.05	0.05	0.05	0.05	0.05
	200	0.05	0.05	0.05	0.05	0.05	0.05
$\kappa=0.50$	20	0.05	0.05	0.04	0.04	0.04	0.04
	40	0.05	0.05	0.04	0.05	0.05	0.05
	60	0.05	0.05	0.05	0.05	0.05	0.05
	100	0.05	0.05	0.04	0.05	0.05	0.05
	200	0.05	0.05	0.05	0.05	0.05	0.05
(b) Exact size of the statistic $t^{NW}_{\hat{\alpha}_h(K)}$ using critical values from the N(0,1) at nominal size 5%							
$\kappa=0.05$	20	0.06	0.06	0.07	0.08	0.07	0.09
	40	0.11	0.11	0.13	0.15	0.15	0.19
	60	0.11	0.11	0.14	0.15	0.16	0.19
	100	0.12	0.11	0.15	0.16	0.17	0.21
	200	0.12	0.12	0.15	0.17	0.17	0.21
$\kappa=0.25$	20	0.22	0.23	0.25	0.28	0.28	0.34
	40	0.22	0.24	0.26	0.29	0.30	0.35
	60	0.22	0.24	0.26	0.29	0.29	0.34
	100	0.22	0.23	0.26	0.29	0.30	0.35
	200	0.22	0.23	0.26	0.29	0.30	0.35
$\kappa=0.50$	20	0.32	0.35	0.37	0.40	0.41	0.45
	40	0.33	0.36	0.38	0.41	0.43	0.46
	60	0.34	0.37	0.38	0.41	0.43	0.45
	100	0.33	0.35	0.37	0.40	0.42	0.45
	200	0.34	0.37	0.39	0.41	0.43	0.46

The limit distributions of $K^{-1/2}t^{\mathrm{STD}}_{\alpha_h(K)}$ have different features (see Table 5). In particular, the effect of a change in κ is not uniform; it varies according to the value of c for a given δ. When $c<0$, an increase in κ reduces the spread and the importance of the tails. However, when $c=0$, the tails become longer as κ increases within the range $0.25 \le \kappa < 0.50$ and become shorter with further increases in κ.

To assess the quality of the approximations to the finite sample distributions, we present in Tables 6 and 7 the exact size of tests based on the asymptotic critical values for a nominal size of 5%.[2] In general, the limit distribution is a good approximation for the statistics $t^{\mathrm{NW}}_{\alpha_h(K)}$ and $K^{-1/2}t^{\mathrm{STD}}_{\alpha_h(K)}$ even for relatively small samples. The approach of the finite

[2] We also considered the case with a nominal size of 10%. The qualitative outcome is the same, hence the results are omitted.

Table 7
Exact size of the test $K^{-1/2} t^{\mathrm{STD}}_{\alpha_h(K)}$ using asymptotic critical values at nominal size 5%

	T	$\delta=0.0$		$\delta=-0.5$		$\delta=-0.9$	
		$c=-5$	$c=0$	$c=-5$	$c=0$	$c=-5$	$c=0$
$\kappa=0.01$	20	0.07	0.07	0.07	0.06	0.06	0.06
	40	0.06	0.05	0.06	0.06	0.05	0.05
	80	0.05	0.06	0.05	0.06	0.05	0.06
	160	0.05	0.05	0.05	0.05	0.05	0.05
$\kappa=0.25$	20	0.06	0.05	0.06	0.05	0.06	0.06
	40	0.05	0.05	0.05	0.05	0.05	0.05
	80	0.05	0.05	0.05	0.05	0.05	0.06
	160	0.05	0.05	0.05	0.06	0.05	0.05
$\kappa=0.75$	30	0.06	0.06	0.06	0.06	0.07	0.07
	60	0.05	0.05	0.05	0.05	0.06	0.06
	100	0.05	0.05	0.06	0.06	0.06	0.06
	200	0.05	0.05	0.05	0.05	0.06	0.06

sample distribution to the limit one is faster when $\kappa \geq 0.25$ and slower when κ is small (e.g., $\kappa=0.05$). For example, when $\kappa \geq 0.25$, size distortions are negligeable when $T \geq 40$. For $\kappa=0.05$ and $T=20$ or 40, the distortions are more important and increase as δ increases (in absolute value). For a nominal size of 5% with $T=20$ and $c=-5$, the exact size is 1.4% when $\delta=0$ and 0.6% when $\delta=-0.9$.

To show how the normal distribution is a bad approximation, we present in Table 6.b the exact sizes of the test $t^{\mathrm{NW}}_{\alpha_h(K)}$ using the critical values from the $N(0,1)$ distribution, for a nominal size of 5%. We observe that the exact sizes are far above the nominal one and the discrepancies increase as $|\delta|$ and κ increase. For example, when $T=40$, $c=-5$ and $\delta=0$, the exact size is 11% when $\kappa=0.05$ and 22% when $\kappa=0.25$; the corresponding figures are 15% and 30% when $\delta=-0.9$.

5.2. Finite sample power

We consider the power function of the statistic $t^{\mathrm{NW}}_{\alpha_h(K)}$ under the same conditions as those used to analyze the properties of the statistic $t_{\hat{\alpha}_h}$ taking into account the effect of the parameter κ. The results, presented in Table 8, are of particular interest for two reasons. First, they highlight the joint effect of variations in κ and δ. For a given value of δ, the power under the stated alternative decreases as κ increases; on the other hand, for a given value of κ, the power increases as $|\delta|$ increases when $N \geq 32$. This rate of increase appears more rapid for small values of κ; for example, when $N=64$ and $T=32$, the power is 0.68 (resp., 0.93) for $\delta=0$ (resp., -0.9) when $\kappa=1/16$; on the other hand, it is 0.18 (resp., 0.23) when $\delta=0$ (resp., $\delta=-0.9$) when $\kappa=1/2$. Hence, the joint effect of variations in κ and δ is more important for small values of κ. For relatively small values of κ (e.g., $\kappa \leq 1/8$), we observe properties that are similar to those of the statistic t_{α_h}; indeed, if N is small, the power approaches the nominal size 5% faster as $|\delta|$ increases (for example, with $\kappa=1/16$, $N=8$ and $T=128$, the power is 0.11 when $\delta=0$ and 0.08 when $\delta=-0.9$); the power increases with N (resp., T) if T (resp., N) is fixed. However, when $\kappa \geq 1/4$, we do not observe a clear

Table 8
Power of the statistic $t^{\mathrm{NW}}_{\hat{\alpha}_h(K)}$ ($\alpha_h = \beta(\exp(\gamma h) - 1)/\gamma$, $\beta = 0.1$ $\gamma = -0.02$)

	$N \backslash T$	(a) $\delta = 0.0$							(b) $\delta = -0.5$							(c) $\delta = -0.9$						
		8	16	32	64	128	256	512	8	16	32	64	128	256	512	8	16	32	64	128	256	512
$\kappa = 1/16$	8		0.12	0.11	0.11	0.11	0.12	0.12		0.10	0.11	0.09	0.10	0.11	0.09		0.09	0.09	0.08	0.08	0.08	0.08
	16		0.23	0.20	0.22	0.21	0.22	0.22		0.20	0.18	0.19	0.18	0.19	0.20		0.17	0.17	0.16	0.15	0.15	0.16
	32		0.44	0.42	0.40	0.41	0.44	0.43		0.48	0.44	0.42	0.42	0.44	0.42		0.60	0.51	0.48	0.46	0.49	0.48
	64		0.67	0.68	0.67	0.69	0.70	0.69		0.79	0.77	0.77	0.79	0.76	0.78		0.95	0.93	0.94	0.93	0.92	0.93
	128		0.87	0.88	0.88	0.87	0.89	0.89		0.96	0.96	0.96	0.96	0.95	0.96		1.00	1.00	1.00	1.00	1.00	1.00
	256		0.95	0.97	0.97	0.97	0.97	0.97		0.99	1.00	1.00	1.00	1.00	1.00		1.00	1.00	1.00	1.00	1.00	1.00
	512		0.96	0.98	0.99	0.99	0.99	0.99		0.99	1.00	1.00	1.00	1.00	1.00		1.00	1.00	1.00	1.00	1.00	1.00
$\kappa = 1/8$	8	0.11	0.10	0.11	0.10	0.10	0.10	0.12	0.11	0.09	0.09	0.08	0.08	0.10	0.09	0.10	0.08	0.09	0.09	0.09	0.08	0.08
	16	0.18	0.20	0.18	0.19	0.16	0.19	0.19	0.18	0.18	0.17	0.16	0.15	0.17	0.17	0.17	0.15	0.16	0.14	0.14	0.14	0.16
	32	0.37	0.37	0.37	0.35	0.35	0.32	0.35	0.42	0.39	0.38	0.38	0.38	0.38	0.36	0.56	0.47	0.44	0.40	0.42	0.43	0.40
	64	0.54	0.58	0.57	0.57	0.58	0.57	0.55	0.69	0.69	0.68	0.67	0.70	0.67	0.67	0.92	0.90	0.90	0.89	0.89	0.88	0.87
	128	0.71	0.73	0.76	0.73	0.71	0.73	0.75	0.85	0.85	0.87	0.85	0.87	0.86	0.86	0.99	0.99	0.99	0.99	0.99	1.00	0.99
	256	0.72	0.76	0.82	0.79	0.78	0.77	0.80	0.87	0.89	0.93	0.90	0.91	0.90	0.90	0.98	0.99	1.00	0.99	1.00	1.00	1.00
	512	0.58	0.65	0.73	0.72	0.75	0.73	0.77	0.72	0.77	0.85	0.84	0.85	0.85	0.87	0.86	0.90	0.96	0.97	0.96	0.96	0.97
$\kappa = 1/4$	8	0.09	0.10	0.09	0.09	0.08	0.10	0.10	0.08	0.09	0.10	0.09	0.08	0.10	0.09	0.08	0.09	0.09	0.08	0.08	0.08	0.08
	16	0.15	0.17	0.14	0.15	0.13	0.16	0.16	0.15	0.16	0.15	0.14	0.13	0.15	0.15	0.15	0.15	0.16	0.13	0.14	0.14	0.14
	32	0.26	0.25	0.27	0.25	0.25	0.23	0.26	0.29	0.32	0.29	0.28	0.29	0.27	0.28	0.39	0.39	0.37	0.34	0.37	0.35	0.34
	64	0.35	0.36	0.37	0.35	0.37	0.37	0.34	0.44	0.48	0.46	0.48	0.47	0.46	0.44	0.69	0.73	0.71	0.72	0.72	0.71	0.69
	128	0.39	0.39	0.37	0.36	0.34	0.36	0.36	0.49	0.51	0.48	0.46	0.45	0.46	0.48	0.72	0.74	0.75	0.72	0.74	0.71	0.72
	256	0.29	0.31	0.33	0.30	0.28	0.27	0.31	0.36	0.39	0.39	0.37	0.34	0.34	0.36	0.49	0.53	0.54	0.48	0.47	0.44	0.47
	512	0.19	0.22	0.19	0.21	0.20	0.18	0.22	0.23	0.22	0.21	0.21	0.20	0.20	0.25	0.30	0.27	0.23	0.25	0.25	0.21	0.25
$\kappa = 1/2$	8	0.07	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.08	0.08	0.09	0.08	0.07	0.09	0.08	0.08	0.07	0.07	0.08	0.07	0.08
	16	0.12	0.12	0.11	0.12	0.11	0.12	0.11	0.12	0.13	0.12	0.13	0.12	0.12	0.14	0.14	0.15	0.14	0.13	0.13	0.14	0.15
	32	0.14	0.16	0.16	0.16	0.16	0.15	0.15	0.18	0.18	0.18	0.17	0.19	0.16	0.19	0.25	0.24	0.25	0.24	0.26	0.23	0.23
	64	0.17	0.17	0.18	0.18	0.17	0.15	0.16	0.18	0.20	0.19	0.19	0.18	0.18	0.19	0.24	0.26	0.23	0.24	0.24	0.25	0.24
	128	0.13	0.16	0.17	0.14	0.13	0.16	0.15	0.12	0.17	0.17	0.14	0.13	0.15	0.15	0.16	0.18	0.19	0.17	0.16	0.17	0.16
	256	0.10	0.12	0.13	0.11	0.12	0.12	0.11	0.10	0.12	0.13	0.12	0.12	0.12	0.11	0.10	0.15	0.15	0.12	0.14	0.14	0.13
	512	0.09	0.10	0.08	0.10	0.09	0.08	0.11	0.10	0.10	0.08	0.10	0.09	0.09	0.11	0.11	0.11	0.09	0.11	0.09	0.09	0.11

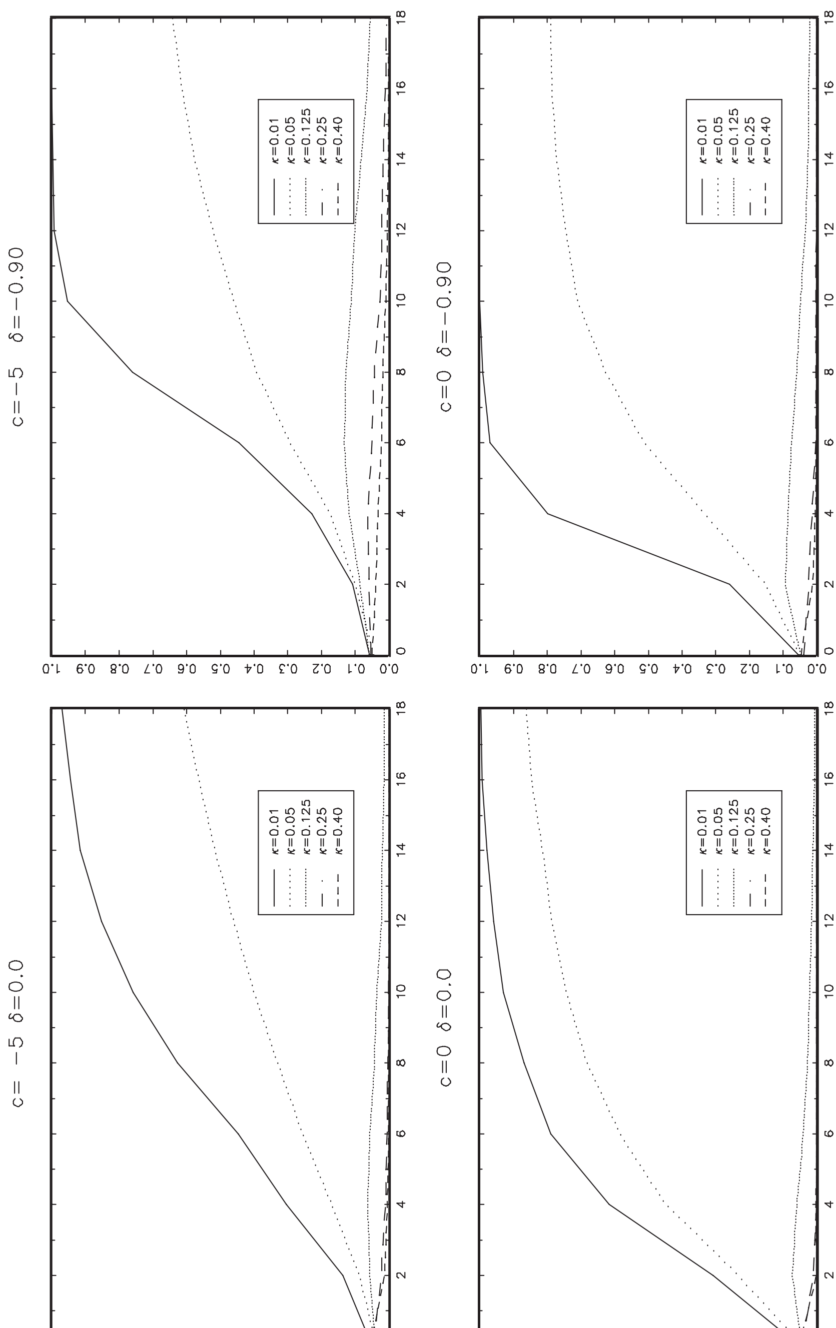

Fig. 2. Asymptotic power of the test $t^{\mathrm{NW}}_{\alpha_{H(K)}}$ under alternative H_1.

monotonic behavior of the power function with either N or T fixed. It remains, nevertheless, a clear property that power is more sensitive to increases in the span N than to increases in the number of observations T, as was the case for the statistic $t_{\hat{\alpha}_h}$.

The second feature of interest in the results of Table 8 is to assess the importance of using K-periods returns instead of a single period by comparing these results with those of Table 3, in particular for $\kappa = 1/16$. The feature that stands out is the fact that for any pair (N,T), the power of the statistic $t_{\hat{\alpha}_h}$ is greater than that of $t^{\mathrm{NW}}_{\alpha_h(K)}$. However, the differences are reduced considerably when $|\delta|$ increases (see, e.g., $\delta = -0.90$) and $N \geq 128$. This suggests that there is little advantage in using K-periods returns.

5.3. Asymptotic power

To study the asymptotic power function of the test $t^{\mathrm{NW}}_{\alpha_h(K)}$, we simulate the limiting distribution (15) and evaluate the rejection frequency (using the asymptotic critical values under the null hypothesis) with 2000 replications for a range of values for the parameters $g = \beta N \sqrt{\tau}$, c, κ and δ. Fig. 2 presents the power function of $t^{\mathrm{NW}}_{\alpha_h(K)}$ for a test with nominal size 5% against one-sided alternatives for $\kappa = 0.01$, 0.05, 0.125, 0.25 and 0.40. The results indicate a strong influence on the power function of various parameters, especially κ. Indeed, for any fixed pair (c,δ), the power decreases as κ increases for all values $g > 0$.

To summarize, the asymptotic framework adopted in the context of K-periods returns of length h shows that the limit distribution of the statistic $t^{\mathrm{NW}}_{\alpha_h(K)}$ so obtained is, for any κ, a good approximation to the exact distribution for values of the sample size T as small as 60.

6. Implications for the empirical results of Fama and French (1988b)

Fama and French (1988b) have estimated, for different values of K, the parameter $\alpha_h(K)$, with a constant added to regression (9). The data pertains to prices of selected portfolios from firms listed on the New York Stock Exchange for the period 1927–1986 which were used to construct returns using first-differences of their logarithmic value. The monthly, quarterly and annual returns were obtained using non-overlapping observations (hence, avoiding the problem of serial correlation in theresiduals of regression (9)) while returns for horizons of 2–4 years were constructed using overlapping observations, in which case the approach of Hansen and Hodrick (1980) was used to adjust the standard errors.

In this section, we analyze the implications of our theoretical results for the empirical results presented in Tables 3 and 4 of Fama and French. To this effect, we have first computed the relevant correlation coefficient δ and the autoregressive coefficient ϕ of an AR(1) process for the dividend–price ratio based on results presented in their Tables 2 and 6. Secondly, we have computed, using 10,000 simulations, the asymptotic p-values of the t-statistics reported by Fama and French using the asymptotic distributions (7) and (17) appropriately modified for the inclusion of a constant in the regression.

Table 9 presents a summary of relevant results from Fama and French (1988b) for real returns of equal-weighted and value-weighted portfolios. They show that, using critical values based on the normal asymptotic approximation, the null hypothesis of market

Table 9
Empirical results related to Fama and French (1988b)

Period	Value-weighted portfolio			Equal-weighted portfolio		
	b	t_b	p-value	b	t_b	p-value
1927–1986 (T= 60)						
	$\phi=0.81$	$\delta=-0.79$		$\phi=0.78$	$\delta=-0.61$	
$K=1$	5.32	2.35	0.01	5.48	2.04	0.02
$K=2$	9.08	2.31	0.03	10.06	2.05	0.05
$K=3$	11.73	2.51	0.03	12.38	2.02	0.04
$K=4$	13.44	2.46	0.03	12.64	1.86	0.07
1927–1956 (T= 30)						
	$\phi=0.64$	$\delta=-0.63$		$\phi=0.79$	$\delta=-0.50$	
$K=1$	9.61	2.16	0.01	4.21	1.02	0.15
$K=2$	19.43	2.65	0.01	10.18	1.28	0.16
$K=3$	24.73	2.74	0.01	12.92	1.23	0.18
$K=4$	23.00	2.21	0.04	9.58	0.84	0.30
1941–1986 (T= 46)						
	$\phi=0.84$	$\delta=-0.87$		$\phi=0.78$	$\delta=-0.89$	
$K=1$	4.40	2.29	0.01	6.99	3.24	0.00
$K=2$	7.21	2.36	0.03	10.89	3.07	0.01
$K=3$	9.66	2.91	0.02	12.37	2.96	0.01
$K=4$	13.34	3.18	0.01	14.19	2.90	0.01

The entries for b and t_b are from Fama and French (1988b, Tables 3 and 4).
Regression: $\sum_{i=1}^{K} R(t+i) = a + bX(t) + \epsilon_K(t)$.
Model: $R(t+1)=\alpha+\beta X(t)+\epsilon(t)$, $X(t)=\phi X(t-1)+v(t)$.
Specifications: $\phi=\exp(c/T)$ and $\mathrm{corr}(\epsilon(t),v(t))=\delta$.

efficiency is rejected at the conventional 5% level for all periods with the exception of the equal-weighted portfolio for the subperiod 1927–1956. Using the asymptotic critical values delivered by the asymptotic framework whereby $K/T \rightarrow \kappa$ as $T \rightarrow \infty$, the conclusions are different for the equal-weighted portfolio. Indeed, the p-values show that, in this case, the null hypothesis cannot be rejected for the periods 1927–1986 and 1927–1956. Of interest, given our theoretical results, is the fact that, in general, the p-values of the tests increase as returns over longer horizons are considered. This is consistent with our result that power decreases as κ increases.

Our conclusions, using a conventional 5% level, do not support the claims of Fama and French (1988b) about the predictive power of the dividend–price ratio on real returns for equal-weighted portfolios (a result similar to that obtained by Richardson and Stock, 1989, for the period 1926–1985). The theoretical framework used suggests to interpret with some caution previous empirical results about mean reversion of stock returns over long horizons.

7. Conclusion

We considered inference in a simple regression with returns as the dependent variable and the price–dividend ratio as the dependent variable when the latter is modelled as a nearly integrated process. We first considered an asymptotic framework whereby these variables are observed over a period of length h, with h converging to 0. We then extended this framework to the case of K-periods returns of length h and considered the limit with $K/T \to \kappa$ and $h \to 0$ when $T \to \infty$. We also analyzed the quality of the asymptotic approximations obtained and the power properties of the statistic $t^{NW}_{\alpha_h(K)}$.

Our results show that, under the null hypothesis of unpredictability of returns, the limit distribution of $t^{NW}_{\alpha_h(K)}$ is nonstandard. The effect of on the limiting distribution is to accentuate the importance of the tails of the distribution. When $\kappa \to 0$, the limit distribution of $t^{NW}_{\alpha_h(K)}$ is a linear combination of a standard normal and the extension of the so-called Dickey–Fuller distribution to the nearly integrated case. Simulations have shown that the limiting distributions obtained are good approximations to the finite sample distributions.

An analysis of the power functions showed that an increase in κ induces a decrease in power contrary to the case for the variance ratio statistic (see Perron and Vodounou, 2000). This result is contrary to that obtained by Campbell (2001) based on Bahadur's (1960) approximate slope analysis which indicates that power should increase with an increase in K. This is evidence of the usefulness of the asymptotic framework we adopted. Finally, it is important to note that our results are conditional on a particular value of the sampling interval h; i.e. given h choosing $\kappa = 0$ or $K = 1$ leads to tests with higher power. However, it remains an open question as to how the power function depends on h. Our framework is not directly suited to provide an answer to this interesting issue.

Acknowledgements

This paper is drawn from Chapter 2 of Vodounou's PhD Dissertation at the Université de Montréal (Vodounou, 1997). We thank two referees and an editor for their useful comments. Perron acknowledges financial support form the Social Sciences and Humanities Research Council of Canada (SSHRC), the Natural Sciences and Engineering Council of Canada (NSERC), and the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche du Québec (FCAR).

Appendix A

Throughout, we make extensive use of the following results, which are now standard.

Lemma A.1. *Let* $u_{th} = (\epsilon^*_{th}, v^*_{th})' = (\epsilon_{th}, v_{th})'/h^{1/2}$ *be i.i.d* $(0, \Sigma)$ *with* Σ *defined by Eq. (2). Then, (a)* $T^{-1/2}\sum_{t=1}^{[Tr]} \epsilon^*_{th} \Rightarrow \sqrt{1-\delta^2}W_1(r) + \delta W_2(r)$; *(b)* $T^{-1/2}\sum_{t=1}^{[Tr]} v^*_{th} \Rightarrow \sqrt{\tau}W_2(r)$ *(jointly) where* $W_1(r)$ *and* $W_2(r)$ *are independent Wiener processes defined on* $C[0,1]$ *and* $\Rightarrow$ *denotes weak convergence in distribution. Let* (R_{th}, X_{th}, e_{th}) *be generated by Eq. (5) and* $(R^*_{th}, X^*_{th}, e^*_{th}) = (R_{th}, X_{th}, e_{th})/h^{1/2}$. *Also, let* $c = \gamma N$ *and* $T = N/h$, *then as* $T \to \infty$ *with*

N fixed, we have (c) $T^{-1}\sum_{t=1}^{T} X^{*}_{(t-1)h}\,\epsilon^{*}_{th} \Rightarrow \sqrt{\tau}\int_0^1 J_2(r)\mathrm{d}W_{12}(r)$; *(d)* $T^{-2}\sum_{t=1}^{T} X^{*2}_{(t-1)h} \Rightarrow \tau\int_0^1 J_2^2(r)\mathrm{d}r$; *(e)* $T^{-1}\sum_{t=1}^{T} \epsilon^{*\,2}_{th} \rightarrow_p 1$ *where* $J_2(r) = \int_0^r \exp(c(r-s))\mathrm{d}W_2(s)$, $W_{12}(r) = \sqrt{1-\delta^2}W_1(r) + \delta W_2(r)$.

Proof of Proposition 1. Under the null hypothesis, we can write the *t*-statistic as:

$$t_{\hat{\alpha}_h} = \frac{Nh^{-1}\alpha_h T^{-2}\sum_{t=1}^{T} X^{*2}_{(t-1)h} + T^{-1}\sum_{t=1}^{T} X^{*}_{(t-1)h}\epsilon^{*}_{th}}{\left[\left(T^{-1}\sum_{t=1}^{T}\epsilon^{*2}_{th}\right)\left(T^{-2}\sum_{l=1}^{T} X^{*2}_{(t-1)h}\right) - T^{-1}\left(T^{-1}\sum_{t=1}^{T} X^{*}_{(t-1)h}\epsilon^{*}_{th}\right)^2\right]^{1/2}}.$$

The results (7) and (8) then follow using Lemma A.1. With *K*-periods returns of length *h*, we can write the OLS estimate of $\alpha_h(K)$ in regression (9) as:

$$\hat{\alpha}_h(K) = \frac{\sum_{t=1}^{T-K+1}\sum_{i=0}^{K-1} X_{(t-1)h}R_{(t+i)h}}{\sum_{t=1}^{T-K+1} X^2_{(t-1)h}} \qquad \square$$

Proof of Lemma 1. To prove part (a), we have:

$$\begin{aligned}
&(T/K)T^{-2}\sum_{t=1}^{T-K+1} X^{*}_{(t-1)h}\,\epsilon^{*}_{th}(K)\\
&\quad = (T/K)\int_0^{\kappa}\int_p^{1+p-\kappa}\left[\int_0^{r-p}\exp(c(r-p-s))\mathrm{d}X_{2T}(s)\right]\mathrm{d}X_{1T}(r)\mathrm{d}p\\
&\quad \Rightarrow(\sqrt{\tau/\kappa})\int_0^{\kappa}\left\{\int_p^{1-\kappa+p} J_2(r-p)\mathrm{d}W_{12}(r)\right\}\mathrm{d}p\\
&\quad \equiv\sqrt{\tau}\int_0^1\left\{\int_{\kappa p}^{1-\kappa+\kappa p} J_2(r-\kappa p)\mathrm{d}W_{12}(r)\right\}\mathrm{d}p
\end{aligned}$$

where $J_2(r-p) = \int_0^{r-p}\exp(c(r-p-s))\mathrm{d}W_2(s)$. For part (b), we have, from Lemma A.1,

$$T^{-2}\sum_{t=1}^{T-K+1} X^{*2}_{(t-1)h} = T^{-1}\sum_{t=1}^{T-K+1}(T^{-1/2}X^{*}_{(t-1)h})^2 \Rightarrow \tau\int_0^{1-\kappa} J_2(r)^2\mathrm{d}r.$$

For part (c), we have

$$T^{-1/2}R^{*}_{[Tr]h}(K) = N\frac{\alpha_h}{h}T^{-1}\sum_{i=0}^{K-1}(T^{-1/2}X^{*}_{([Tr]-1+i)h}) + T^{-1/2}\sum_{i=0}^{K-1}\epsilon^{*}_{([Tr]+i)h}$$

$$\Rightarrow \beta N\sqrt{\tau}\int_r^{r+\kappa}J_2(s)\mathrm{d}s + W_{12}(r+\kappa) - W_{12}(r).$$

For part (d), we have $T^{-3/2}\sum_{i=0}^{K-1}X^{*}_{([Tr]-1+i)h}\Rightarrow\sqrt{\tau}\int_r^{r+\kappa}J_2(s)\mathrm{d}s$, hence:

$$T^{-2}K^{-1}\sum_{t=1}^{T-K+1}\sum_{i=0}^{K-1}X^{*}_{(t-1)h}X^{*}_{(t-1+i)h}$$

$$= (T/K)\int_0^{1-\kappa}[T^{-1/2}X^{*}_{([Tr]-1)h}][T^{-3/2}\sum_{i=0}^{K-1}X^{*}_{([Tr]-1+i)h}]\mathrm{d}r$$

$$\Rightarrow \frac{\tau}{\kappa}\int_0^{1-\kappa}\int_r^{r+\kappa}J_2(r)J_2(s)\mathrm{d}s\mathrm{d}r.$$

For part (e), we have from parts (a) to (c):

$$T^{-1/2}\hat{\xi}^{*}_{[Tr]h}(K) = T^{-1/2}R^{*}_{[Tr]h}(K) - \hat{\alpha}_h(K)T^{-1/2}X^{*}_{([Tr]-1)h}$$

$$\Rightarrow \beta N\sqrt{\tau}\int_r^{r+\kappa}J_2(s)\mathrm{d}s + W_{12}(r+\kappa) - W_{12}(r) - J_2(r)$$

$$\times\frac{\int_0^{\kappa}\left\{\int_p^{1-\kappa+p}J_2(r-p)\mathrm{d}W_{12}(r)\right\}\mathrm{d}p}{\int_0^{1-\kappa}J_2^2(r)\mathrm{d}r}\equiv H_{12}(r)$$

In a similar way, $T^{-1/2}\hat{\xi}^{*}_{([Tr]-[Tj])h}(K)\Rightarrow H_{12}(r-u)$. Hence,

$$T^{-4}\sum_{j=-K+1}^{K-1}\left(1-\frac{|j|}{K}\right)\hat{R}(j)\Rightarrow\tau\int_{-\kappa}^{\kappa}\left(1-\frac{|u|}{\kappa}\right)H(u)\mathrm{d}u$$

with $H(u) = \int_u^{1-\kappa}J_2(r)J_2(r-u)H_{12}(r)H_{12}(r-u)\mathrm{d}r$. Writing

$$\hat{V}(\hat{\alpha}_h(K)) = \frac{T^{-4}\sum_{j=-K+1}^{K-1}\left(1-\frac{|j|}{K}\right)\hat{R}(j)}{\left\{T^{-2}\sum_{t=1}^{T-K+1}X^{*2}_{(t-1)h}\right\}^2},$$

and using the results above, part (e) follows. □

Proof of Eq. (17). It is sufficient to derive the limit of $\hat{V}_H(\hat{\alpha}_h(K))$ which can be written as:

$$\hat{V}_H(\hat{\alpha}_h(K)) = \frac{(T-K+1)^{-1}\sum_{j=-K+1}^{K-1}\left(1-\frac{|j|}{K}\right)[T^{-1}\hat{R}_X^*(j)][T^{-1}\hat{R}_\xi^*(j)]}{\left\{\left(1-\frac{K}{T}+\frac{1}{T}\right)^{-1}T^{-2}\sum_{t=1}^{T-K+1}X_{(t-1)h}^{*2}\right\}^2}$$

It is straightforward to show that for $(j-1)/T \leq u < j/T (j=1,\ldots,T-1)$, $T^{-1}\hat{R}_X^*(j) \Rightarrow (1-\kappa)^{-1}\tau\int_u^{1-\kappa}J_2(r)J_2(r-u)\mathrm{d}r$ and $T^{-1}\hat{R}_\xi^*(j) \Rightarrow (1-\kappa)^{-1}\int_u^{1-\kappa}H_{12}(r)H_{12}(r-u)\mathrm{d}r$. Hence, $\hat{V}_H(\hat{\alpha}_h(K)) \Rightarrow \hat{\hat{V}}/((1-\kappa)\tau)$ where

$$\hat{\hat{V}} = \frac{\int_{-\kappa}^{\kappa}\left(1-\frac{|u|}{\kappa}\right)\left[\int_u^{1-\kappa}J_2(r)J_2(r-u)\mathrm{d}r\right]\left[\int_u^{1-\kappa}H_{12}(r)H_{12}(r-u)\mathrm{d}r\right]\mathrm{d}u}{\left\{\int_0^{1-\kappa}J_2^2(r)\mathrm{d}r\right\}^2}.$$

□

References

Arnold, L., 1974. Stochastic Differential Equations: Theory and Applications Wiley, New York.

Bachelier, L., 1900. Théorie de la spéculation. Annales Scientifiques de l'Ecole Normale Supérieure 17, 21–86.

Bahadur, R.R., 1960. Stochastic comparison of tests. Annals of Mathematical Statistics 31, 276–294.

Bergstrom, A.R., 1984. Continuous time stochastic models and issues of aggregation over time. In: Griliches, Z., Intriligator, M.D. (Eds.), (Eds.), Handbook of Econometrics, vol. 2. North-Holland, Amsterdam, pp. 1146–1210.

Campbell, J.Y., 2001. Why long horizon? A study of power against persistent alternatives. Journal of Empirical Finance 8, 459–491.

Campbell, J.Y., Shiller, R.J., 1988. The dividend–price ratio and expectations of future dividends and discount factors. Review of Financial Studies 1, 195–228.

Elliott, G., Stock, J.H., 1994. Inference in time series regression when the order of integration of a regressor is unknown. Econometric Theory 10, 672–700.

Fama, E.F., 1965. The behavior of stock market prices. Journal of Business 38, 34–105.

Fama, E.F., 1970. Efficient capital markets price: a review of theory and empirical work. Journal of Finance 25, 383–417.

Fama, E.F., 1991. Efficient capital markets: II. Journal of Finance 46, 1575–1617.

Fama, E.F., French, K.R., 1988a. Permanent and temporary components of stock prices. Journal of Political Economy 96, 246–273.

Fama, E.F., French, K.R., 1988b. Dividend yields and expected stock returns. Journal of Financial Economics 22, 3–24.

Geweke, J., 1981. The approximate slope of econometric tests. Econometrica 49, 1427–1442.

Hansen, L.P., Hodrick, R.J., 1980. Forward exchange rates as optimal predictors of future spot rates: an econometric analysis. Journal of Political Economy 88, 829–853.

Lo, A.W., Mackinlay, A.C., 1988. Stock market prices do not follow random walks: evidence from a simple specification test. Review of Financial Studies 1, 41–66.

Newey, W.K., West, K.D., 1987. A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix. Econometrica 55, 703–708.

Perron, P., 1991a. A continuous time approximation to the unstable first-order autoregressive process: the case without an intercept. Econometrica 59, 211–236.

Perron, P., 1991b. A continuous time approximation to the stationary first-order autoregressive model. Econometric Theory 7, 236–252.

Perron, P., 1991c. Test consistency with varying sampling frequency. Econometric Theory 7, 341–368.

Perron, P., Vodounou, C., 2000. The variance ratio test: An analysis of size and power based on a continuous time asymptotic framework. Unpublished manuscript, Department of Economics, Boston University

Phillips, P.C.B., 1987. Towards a unified asymptotic theory for autoregression. Biometrika 74, 535–547.

Poterba, M.J., Summers, L.H., 1988. Mean reversion in stock prices: evidence and implications. Journal of Financial Economics 22, 27–59.

Richardson, M., Smith, T., 1991. Robust power calculations with tests for serial correlation in stock returns. Unpublished manuscript, University of Pennsylvania.

Richardson, M., Stock, J.H., 1989. Drawing inference from statistics based on multiyear asset return. Journal of Financial Economics 25, 323–348.

Rozeff, M., 1984. Dividend yields are equity risk premiums. Journal of Portfolio Management 11, 68–75.

Shiller, R.J., 1984. Stock prices and social dynamics. Brookings Papers on Economic Activity 2, 457–510.

Shiller, R.J., Perron, P., 1985. Testing the random walk hypothesis: power versus frequency of observation. Economics Letters 18, 381–386.

Vodounou, C., 1997. Essais en économetrie théorique et en économétrie de la finance appliqués aux modèles dynamiques. Unpublished PhD Dissertation, Département de Sciences Économiques. Université de Montréal.